Compensator Design Examples

The Plant:

$$T(s) = \frac{T_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)} \tag{4.11}$$

where

$$T_o = 250, \omega_1 = 2\pi(10), \omega_2 = 2\pi(100), \omega_3 = 2\pi(300)$$

Somewhat arbitrarily we can identify three constituent basic transfer functions which when multiplied together form the composite transfer function.

$$T(s) = \underbrace{\frac{T_o}{\left(1+\frac{s}{\omega_1}\right)}}_{T_a(s)} \underbrace{\frac{1}{\left(1+\frac{s}{\omega_2}\right)}}_{T_b(s)} \underbrace{\frac{1}{\left(1+\frac{s}{\omega_3}\right)}}_{T_c(s)} \tag{4.12}$$

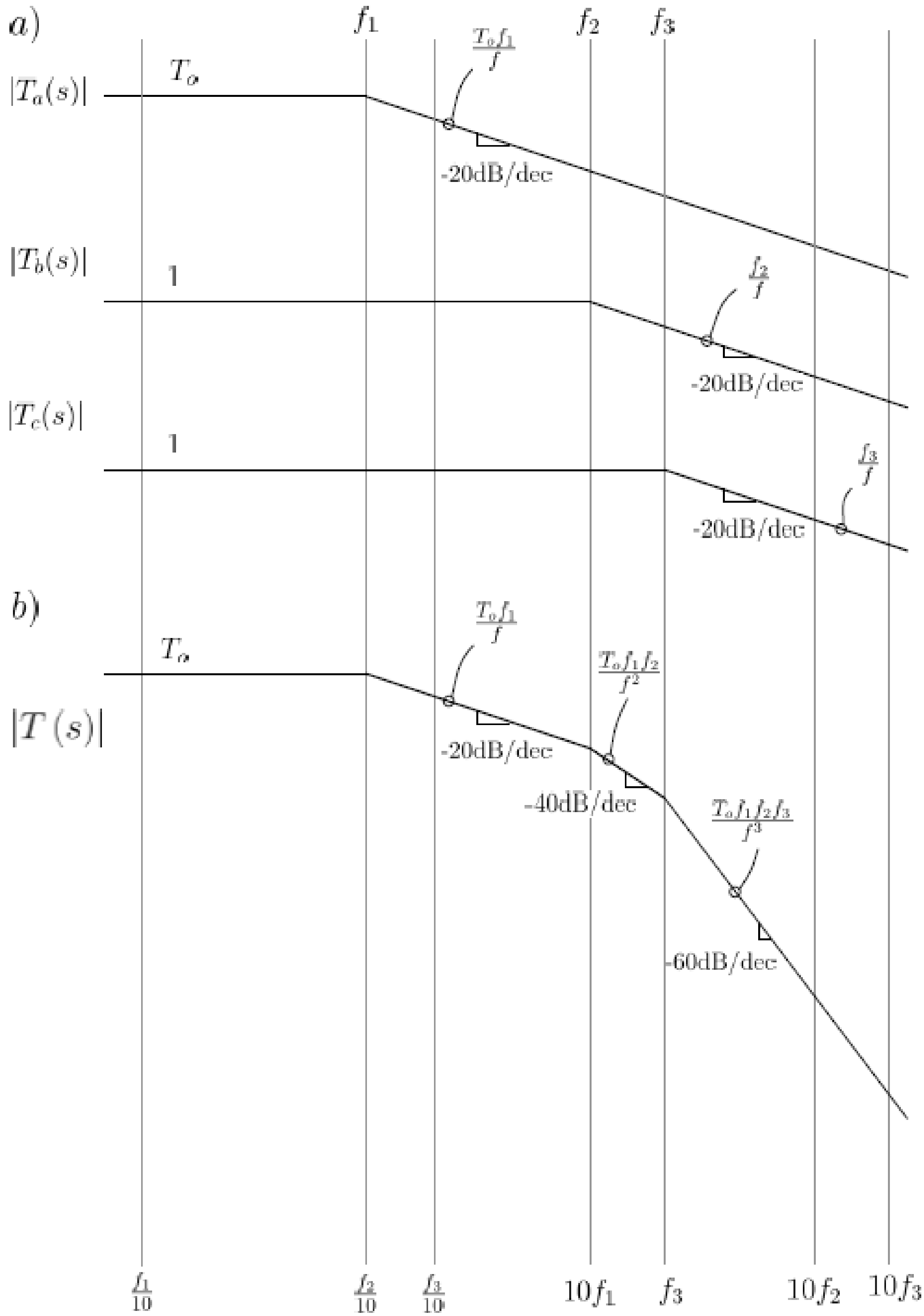

Figure 4.11: a) Asymptotic magnitude plots for the constituent transfer functions, b) Asymptotic magnitude plot for the composite transfer function

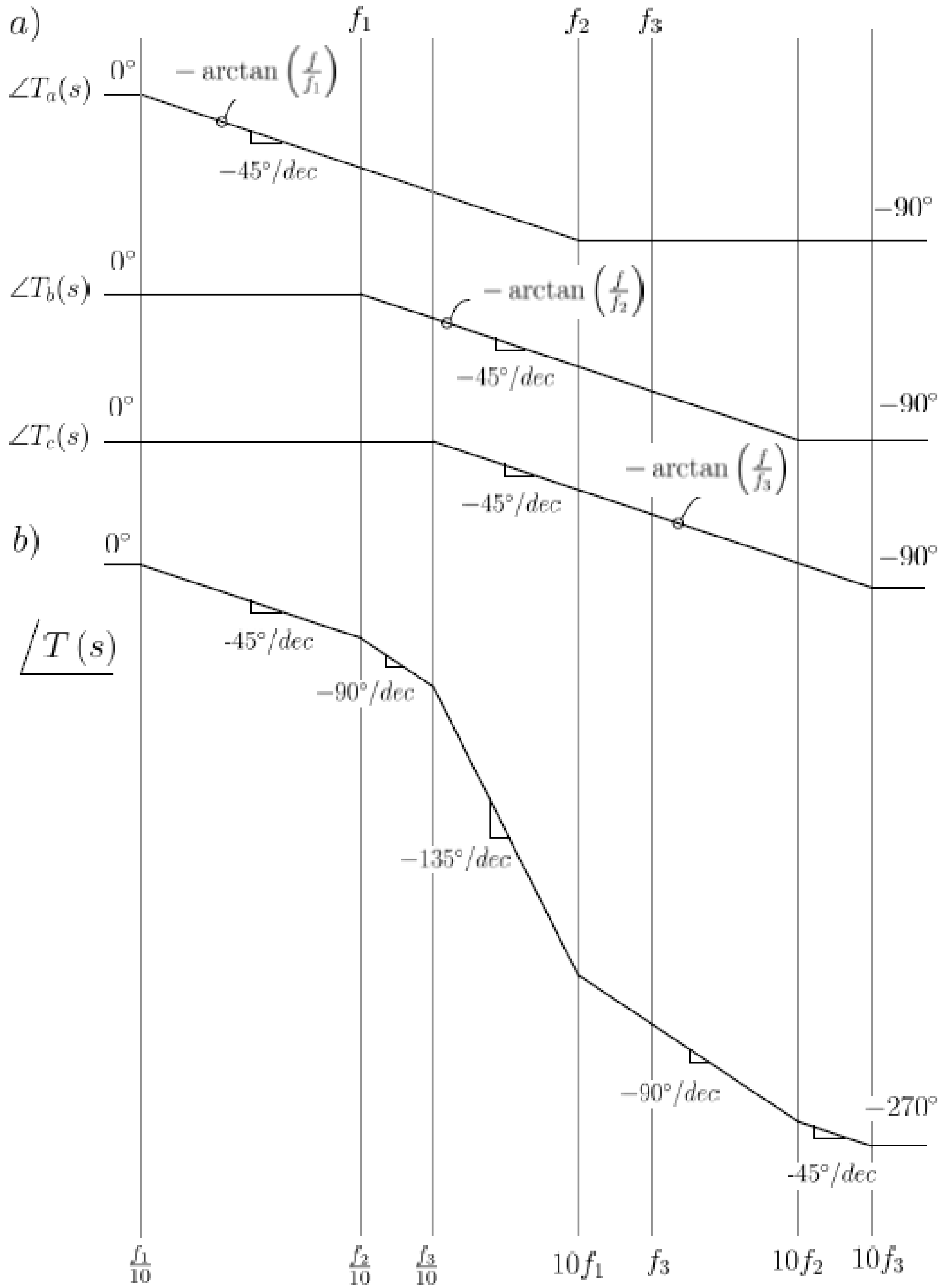

Figure 4.12: a) Asymptotic phase plots for the constituent transfer functions, b) Asymptotic plot plot for the composite transfer function

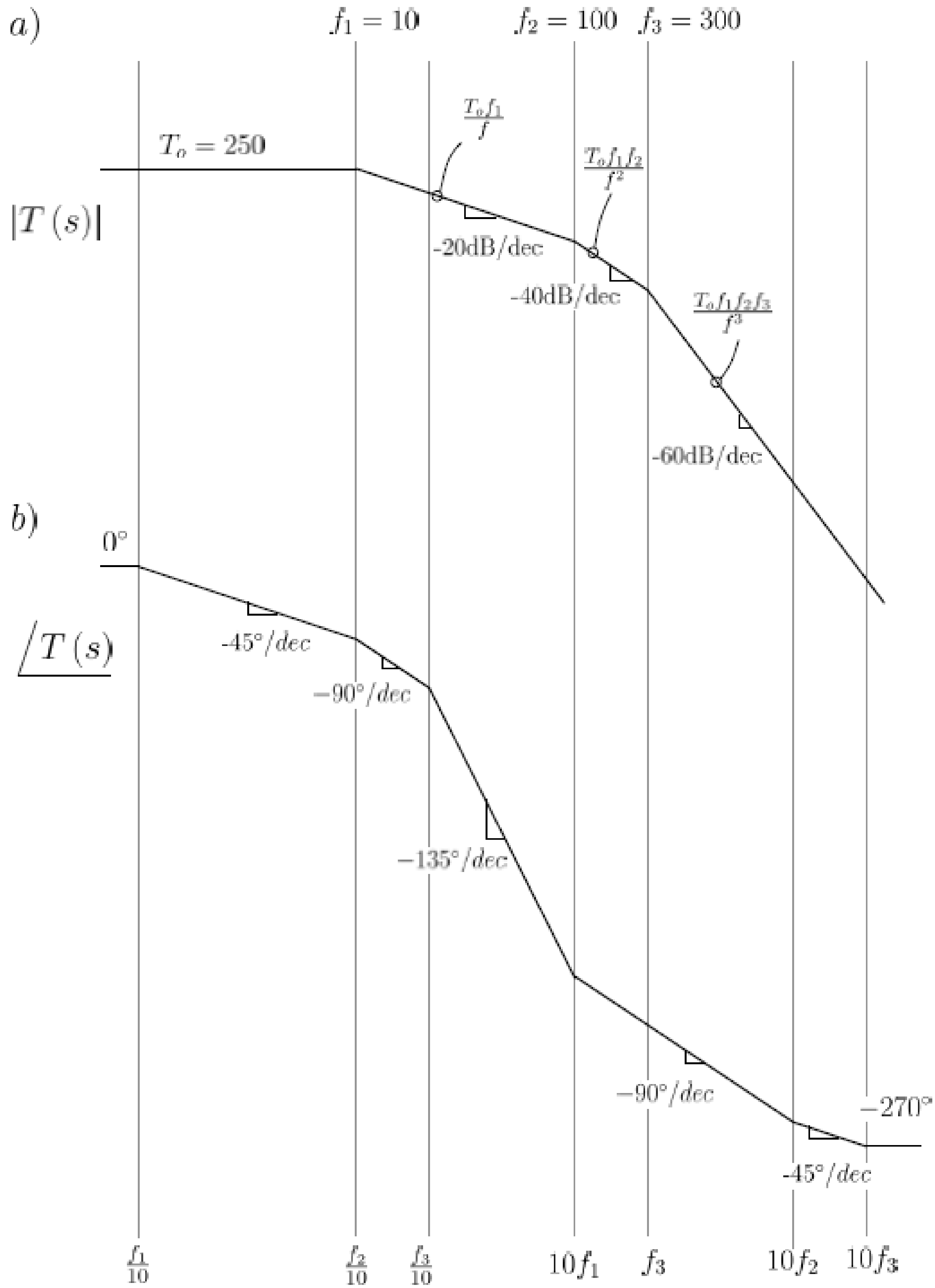

Figure 4.13: Final constructed asymptotic Bode plot showing, a) asymptotic magnitude response, b) asymptotic phase response

Closed loop System:

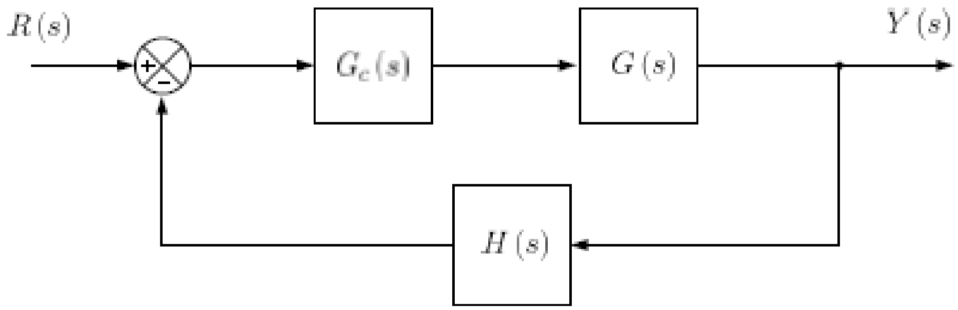

Figure 5.1: Feedback System Block Diagram

To demonstrate the design procedure, in the sequel we will use a plant and feedback gain with the following transfer functions:

$$G(s) = \frac{G_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)} \tag{5.1}$$

$$H(s) = k \tag{5.2}$$

where $G_o = 500$, $\omega_1 = 2\pi(10)$, $\omega_2 = 2\pi(100)$, $\omega_3 = 2\pi(300)$, and $k = 0.5$.

Uncompensated System:

We start our evaluation with the uncompensated loop gain $T(s) = kG_c(s)\,G(s)$, where $G_c(s) = 1$. The loop gain is given as

$$T(s) = \frac{T_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)} \tag{5.8}$$

where

$$T_o = G_o k = 500 \cdot 0.5 = 250$$

Uncompensated Loop Gain:

$$\omega_1 = 2\pi\,(10)\,, \omega_2 = 2\pi\,(100)\,, \omega_3 = 2\pi\,(300)$$

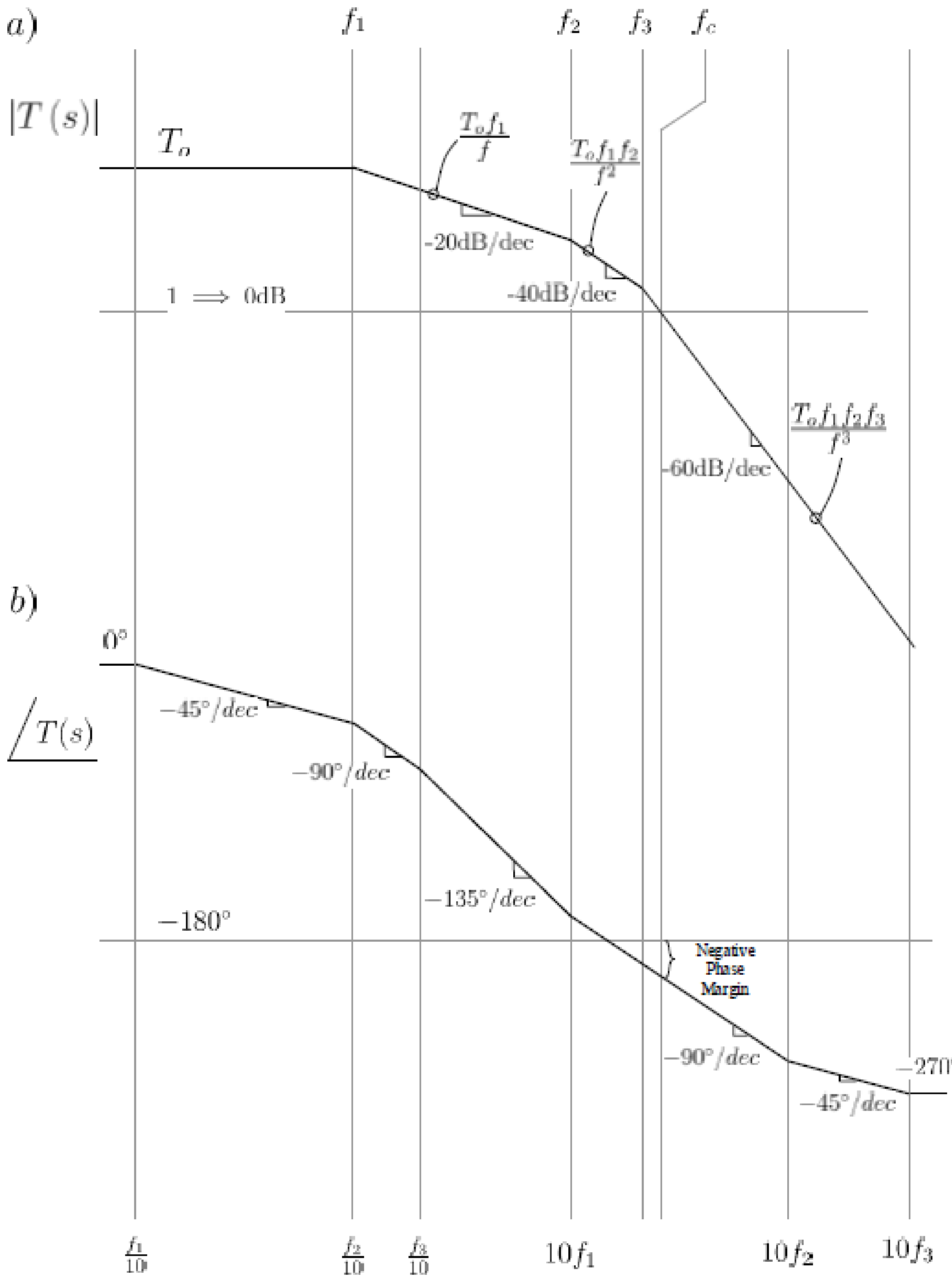

Figure 5.2: Bode Plot: Uncompensated System

f_c, the unity gain crossover frequency, and PM, the phase margin:

$$\frac{T_o f_1 f_2 f_3}{f_c^3} = 1 \implies f_c = \sqrt[3]{T_o f_1 f_2 f_3} \tag{5.9}$$

$$PM = 180 - \arctan\left(\frac{f_c}{f_1}\right) - \arctan\left(\frac{f_c}{f_2}\right) - \arctan\left(\frac{f_c}{f_3}\right) \tag{5.10}$$

➔

result in $f_c = 422$ Hz and $PM = -40°$.

In a similar fashion we can also determine f_{GM}, the frequency at which the phase reaches $-180°$, and subsequently the gain margin:

$$-180 = -\arctan\left(\frac{f_{GM}}{f_1}\right) - \arctan\left(\frac{f_{GM}}{f_2}\right) - \arctan\left(\frac{f_{GM}}{f_3}\right) \tag{5.11}$$

$$GM = -20\log\left(\frac{T_o f_1 f_2}{f_{GM}^2}\right) \tag{5.12}$$

➔

results in $f_{GM} = 184$ Hz and $GM = -17.3$ dB.

Using the Matlab *margin* command:

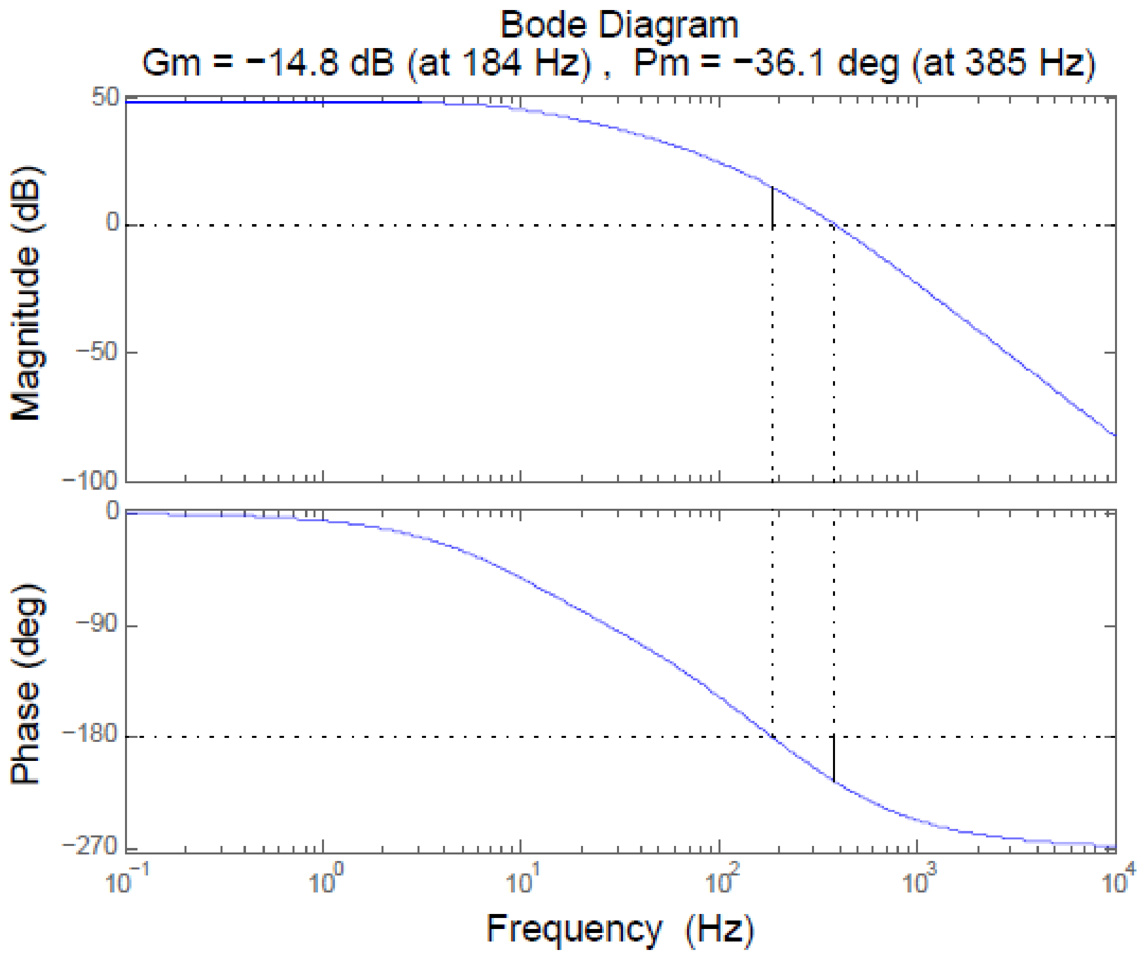

Figure 5.3: Matlab Analysis of Uncompensated System

Table 5.1: Uncompensated System margins

	PM (°)	f_C **(Hz)**	**GM (dB)**	f_{GM} **(Hz)**
Asymptotes	-40	422	-17.3	184
Matlab	-36.1	385	-14.8	184

Negative PM → unstable closed loop system:

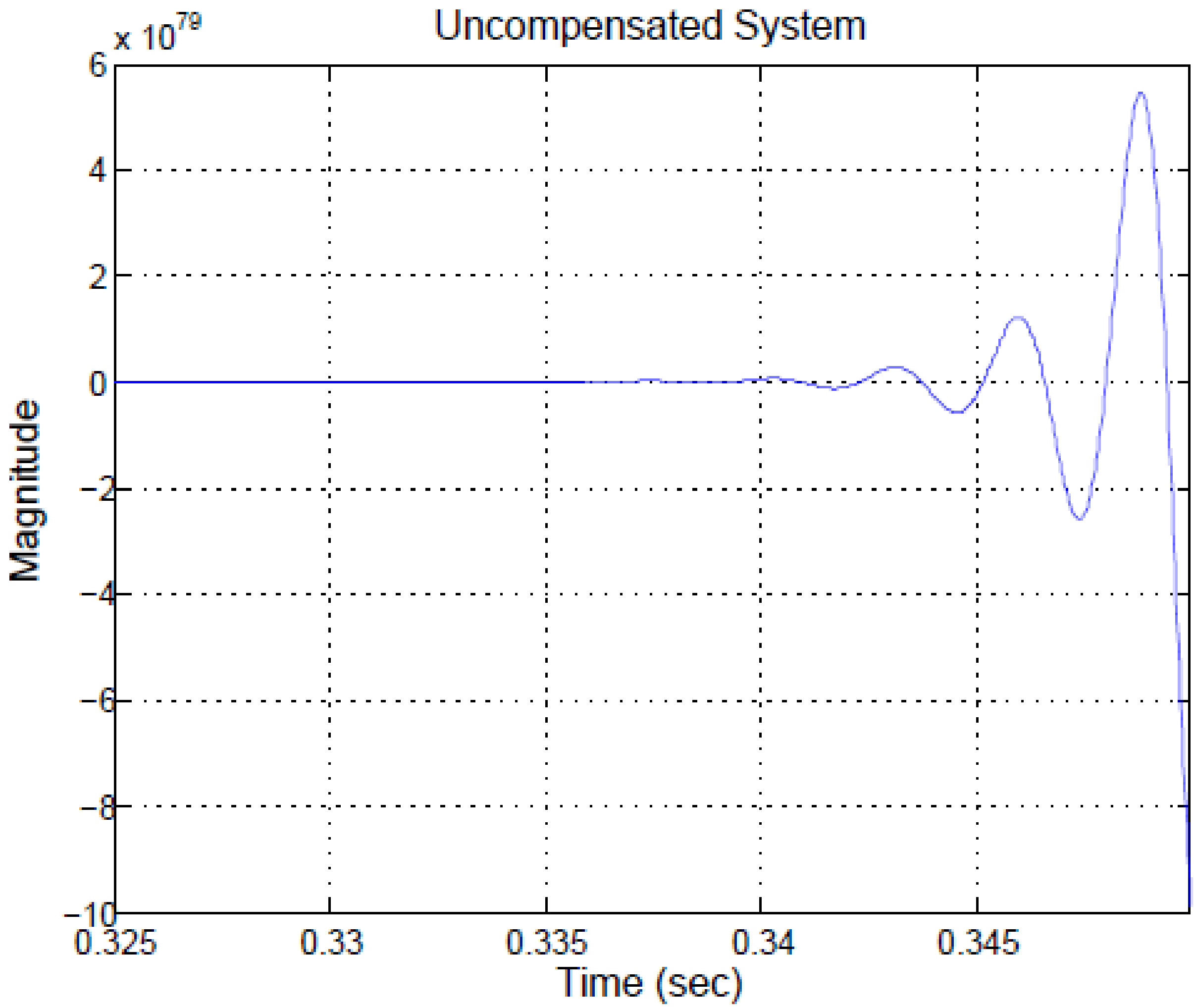

Figure 5.4: Matlab Analysis of Uncompensated System

Compensators considered:

1) Proportional (P) compensator:

$$G_c(s) = k_p \tag{5.3}$$

2) Dominant pole (I, integrator) compensator:

$$G_c(s) = \frac{\omega_I}{s} \tag{5.4}$$

3) Dominant pole with zero (PI, proportional plus integrator) compensator:

$$G_c(s) = \frac{\omega_I}{s}\left(1 + \frac{s}{\omega_z}\right) \tag{5.5}$$

4) Lead compensator:

$$G_c(s) = G_{c_o}\frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_p}}, \qquad \omega_z < \omega_p \tag{5.6}$$

5) Lead with integrator and zero compensator

$$G_c(s) = \frac{\omega_I\left(1 + \frac{s}{\omega_{z_1}}\right)\left(1 + \frac{s}{\omega_{z_2}}\right)}{s\left(1 + \frac{s}{\omega_p}\right)} \tag{5.7}$$

Compensator Design #1: Proportional Compensator

$$G_c(s) = k_p$$

k_p simply represents a constant gain. Note that the effect of varying the value of k_p is to raise and lower the magnitude Bode plot while keeping the phase plot unaffected. So the value of k_p can be set to obtain a unity gain crossover frequency (f_c) which results in an acceptable phase margin.

As a general rule of thumb, to obtain an acceptable phase margin (generally $45° \leq PM \leq 60°$) one usually sets the unity gain crossover frequency (f_c) to occur in the segment of the asymptotic magnitude plot that has a slope of -20dB/dec. From the constructed magnitude plot we find

$$\frac{k_p T_o f_1}{f_c} = 1 \implies f_c = k_p T_o f_1 \tag{5.14}$$

➔

$$\begin{aligned} PM &= 180 - \arctan\left(\frac{f_c}{f_1}\right) - \arctan\left(\frac{f_c}{f_2}\right) - \arctan\left(\frac{f_c}{f_3}\right) \\ &= 180 - \arctan(k_p T_o) - \arctan\left(\frac{k_p T_o f_1}{f_2}\right) - \arctan\left(\frac{k_p T_o f_1}{f_3}\right) \end{aligned} \tag{5.15}$$

With a desired value of phase margin of $PM = 45°$

➔

$$k_p = 0.0311 \text{ and } f_c = 77.65$$

Verify with Matlab *margin* command:

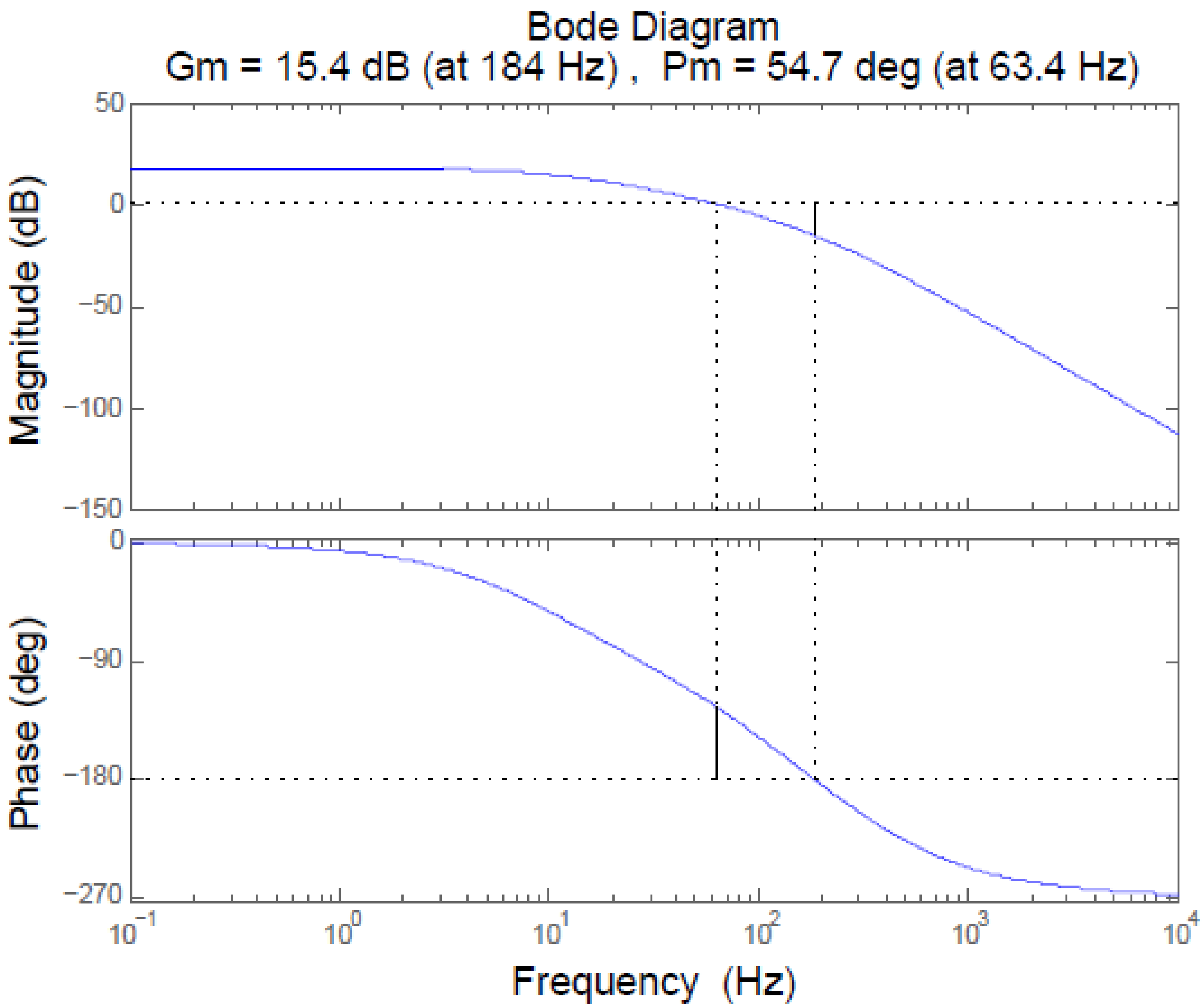

Figure 5.5: Matlab Proportional Compensated Loop Gain Bode Plot

Transient performance:

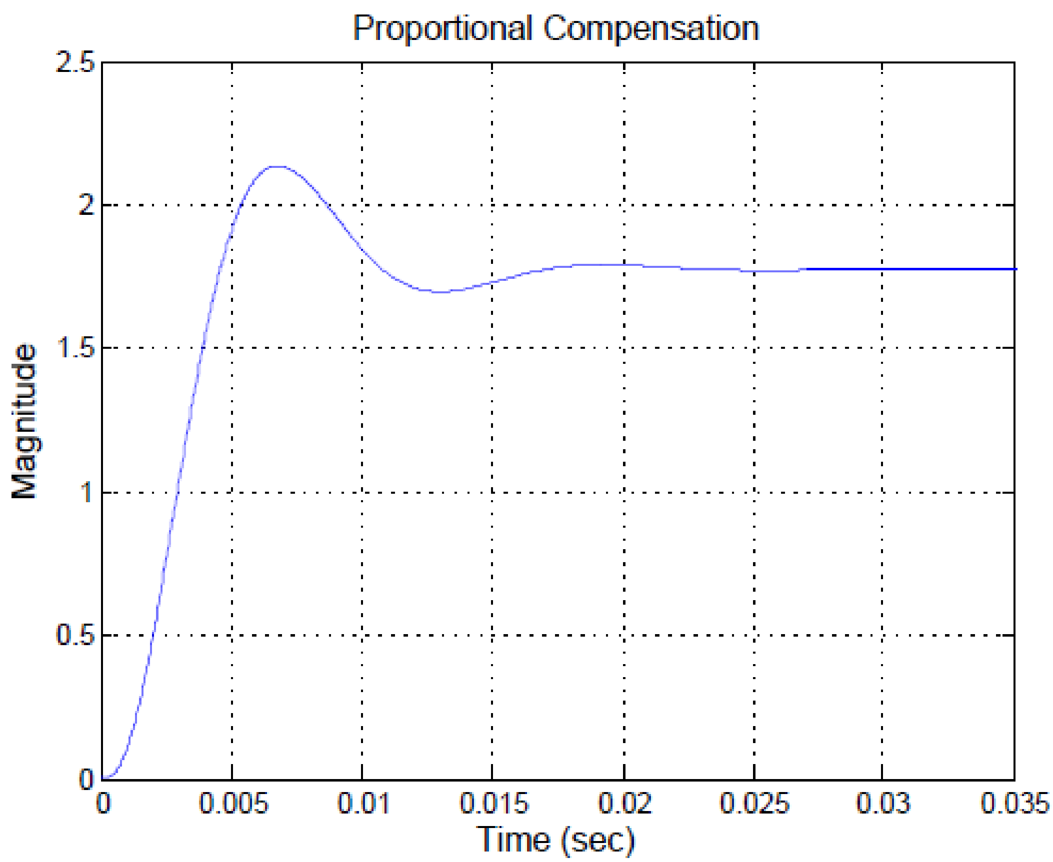

Figure 5.6: Step Response of Proportional Compensated Closed-Loop System

Table 5.2: Proportionally Compensated System Performance Features

Proportional Compensation	
Characteristics	Value
Overshoot	20 %
Rise time	2.9 ms
Settling time	15.4 ms
Steady-state error	−11 %
Bandwidth	63 Hz
Phase margin	55°
Gain margin	15 dB

Compensator Design #2: Dominant Pole Compensator

$$G_c(s) = \frac{\omega_I}{s}$$

where $\omega_I = 2\pi \cdot f_I$

In this case the pole is at zero frequency and so the transfer function is that of an integrator.

f_I is the design parameter, and represents the frequency at which the gain of the integrator is unity.

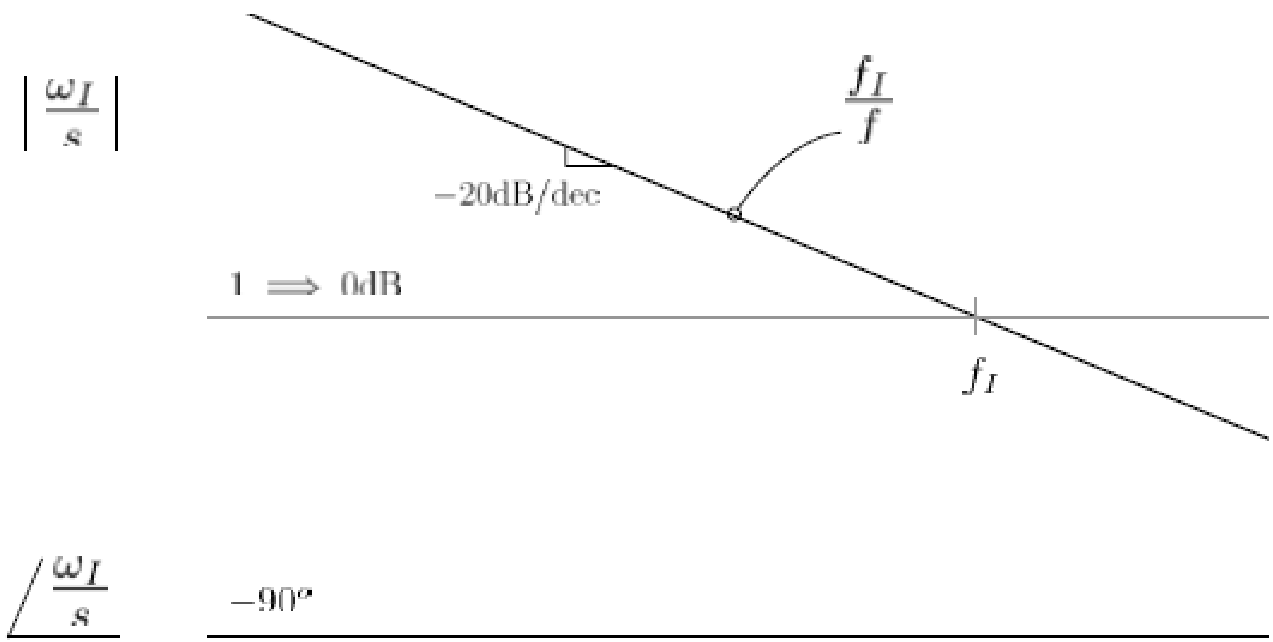

Figure 5.7: Bode Plot: Dominant Pole Compensator

Modified loop gain magnitude response:

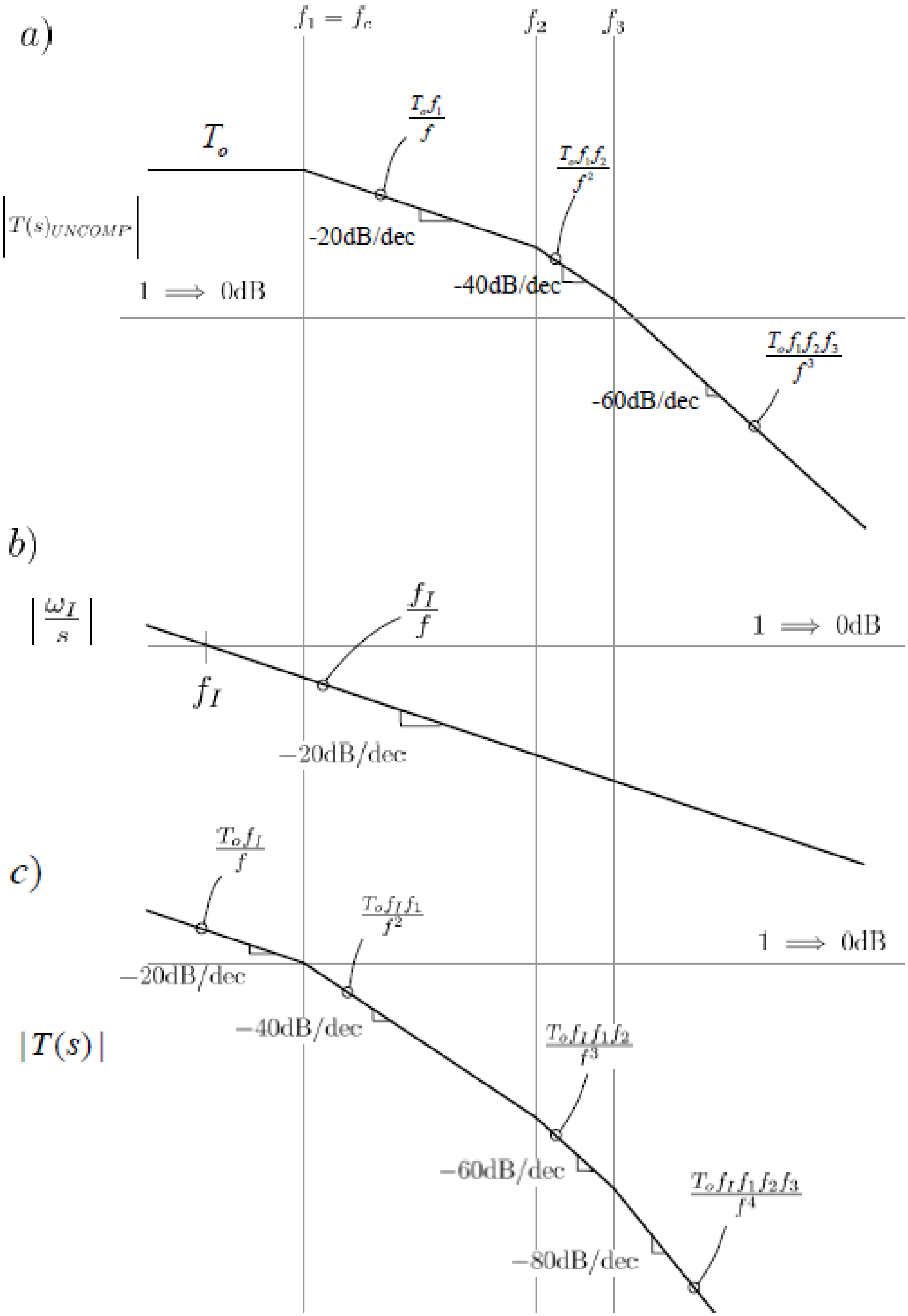

Figure 5.9: Dominant Pole: Magnitude Construction

Modified loop gain phase response:

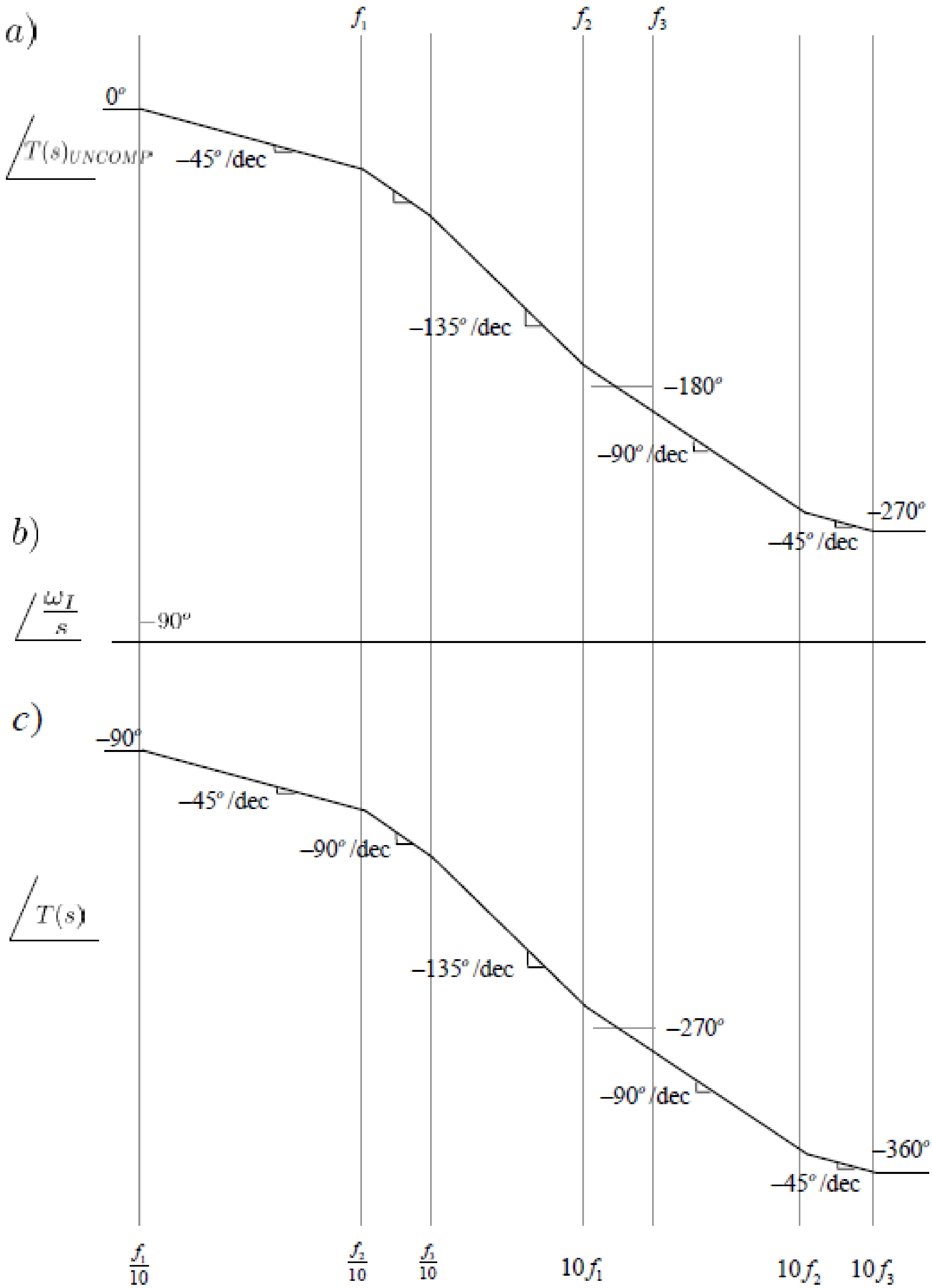

Figure 5.8: Dominant Pole: Phase Construction

Together (magnitude and phase):

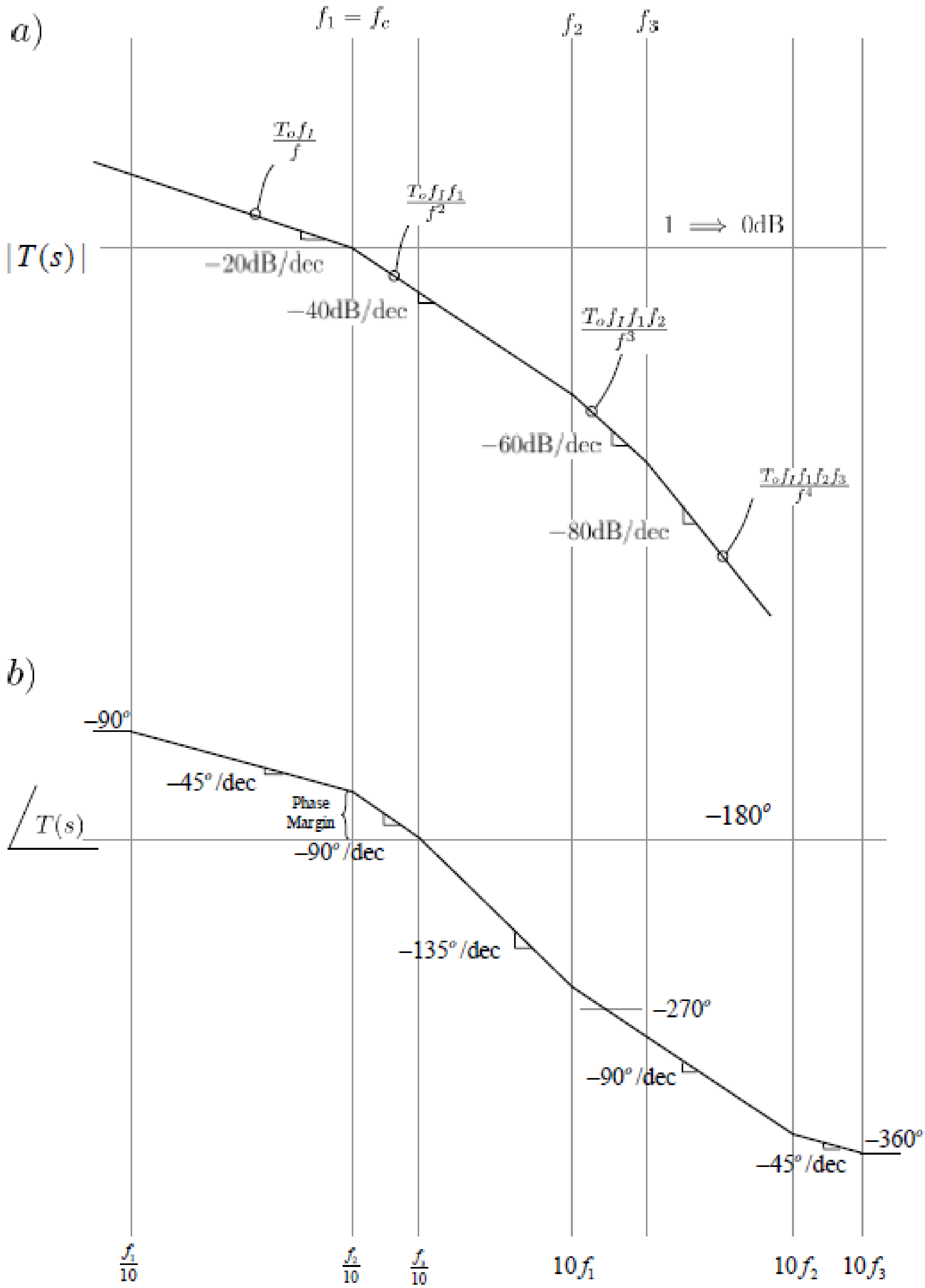

Figure 5.10: Dominant Pole Compensated System

Setting $f_c = f_1$ results in a $PM = +45°$

From the magnitude response we see:

$$\frac{f_I T_o}{f_1} = 1$$

➔

$$f_I = \frac{f_1}{T_o} = \frac{10}{250} = 0.04$$

➔

$$G_c\left(s\right) = \frac{\omega_I}{s} = \frac{2\pi \cdot 0.04}{s}$$

Verify with Matlab *margin* command:

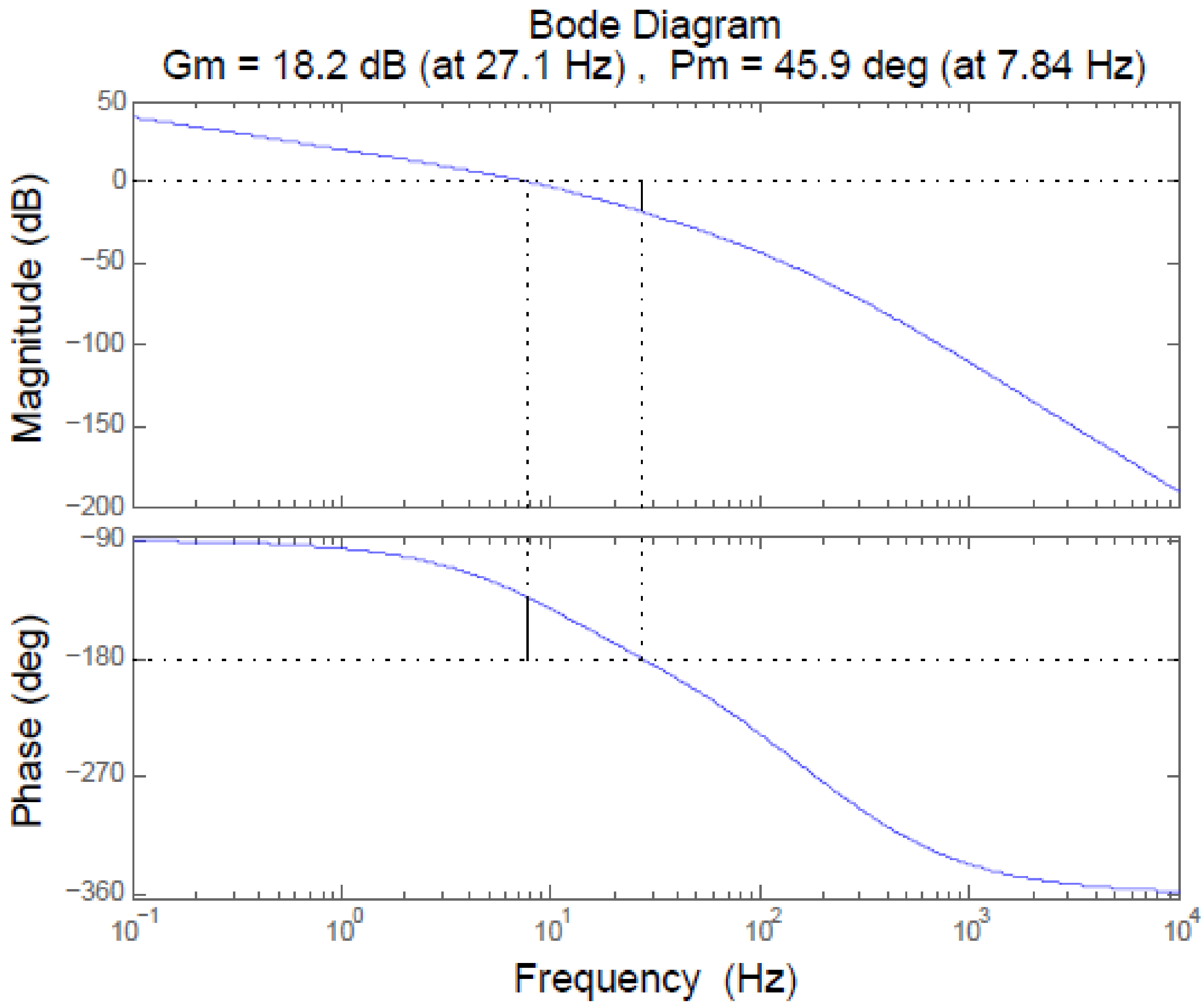

Figure 5.11: Matlab Analysis of Dominant Pole Compensated System

Transient performance:

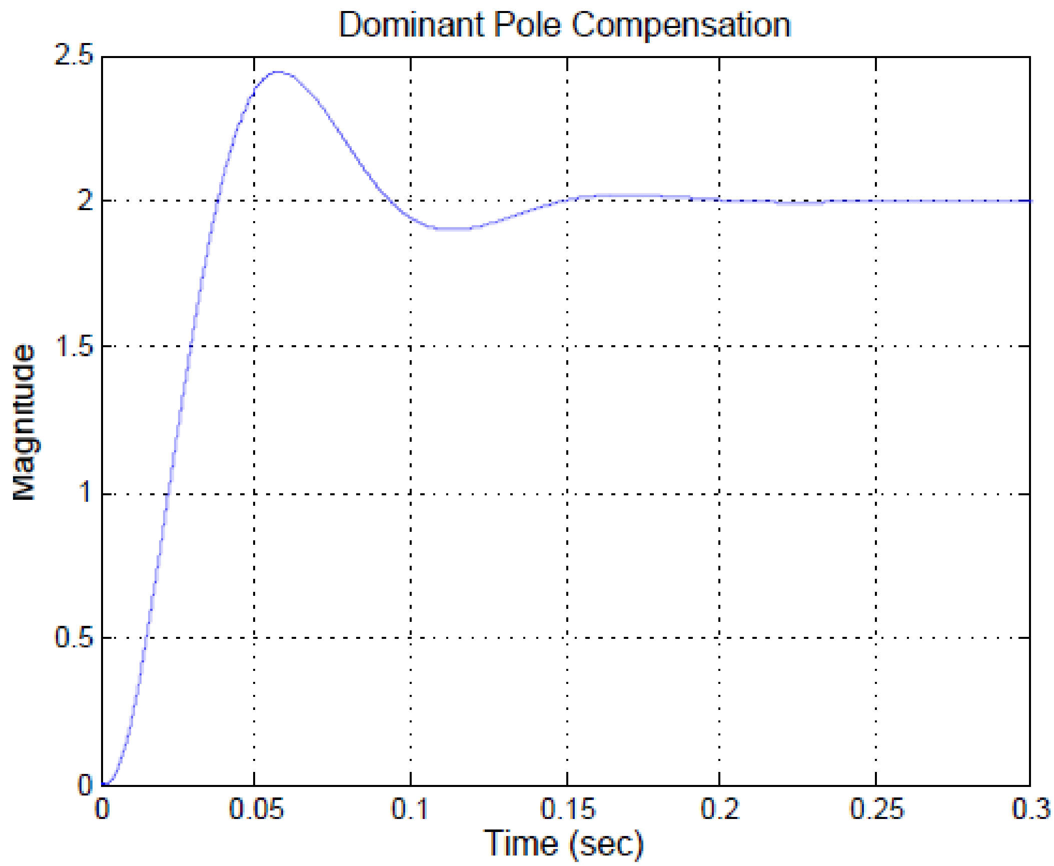

Figure 5.12: Step Response of the Dominant Pole Compensated System

Table 5.3: Dominant Pole Compensated System Features

Dominant Pole Compensation	
Characteristics	Value
Peak amplitude	22.2% overshoot
Rise time	24.7 ms
Settling time	134 ms
Steady-state error	0 %
Bandwidth	7.84 Hz
Phase margin	45.9°
Gain margin	18.2 dB

Compensator Design #3: Dominant Pole with Zero Compensator

$$G_c(s) = \frac{\omega_I}{s}\left(1 + \frac{s}{\omega_z}\right)$$

We will set $\omega_z = \omega_1$**, i.e. the zero will cancel the lowest plant pole**

$$\text{compensated loop gain } T(s) = kG_c(s)G(s)$$

➔

$$T(s) = \frac{T_o \omega_I}{s\left(1 + \frac{s}{\omega_2}\right)\left(1 + \frac{s}{\omega_3}\right)}$$

Modified loop gain magnitude response:

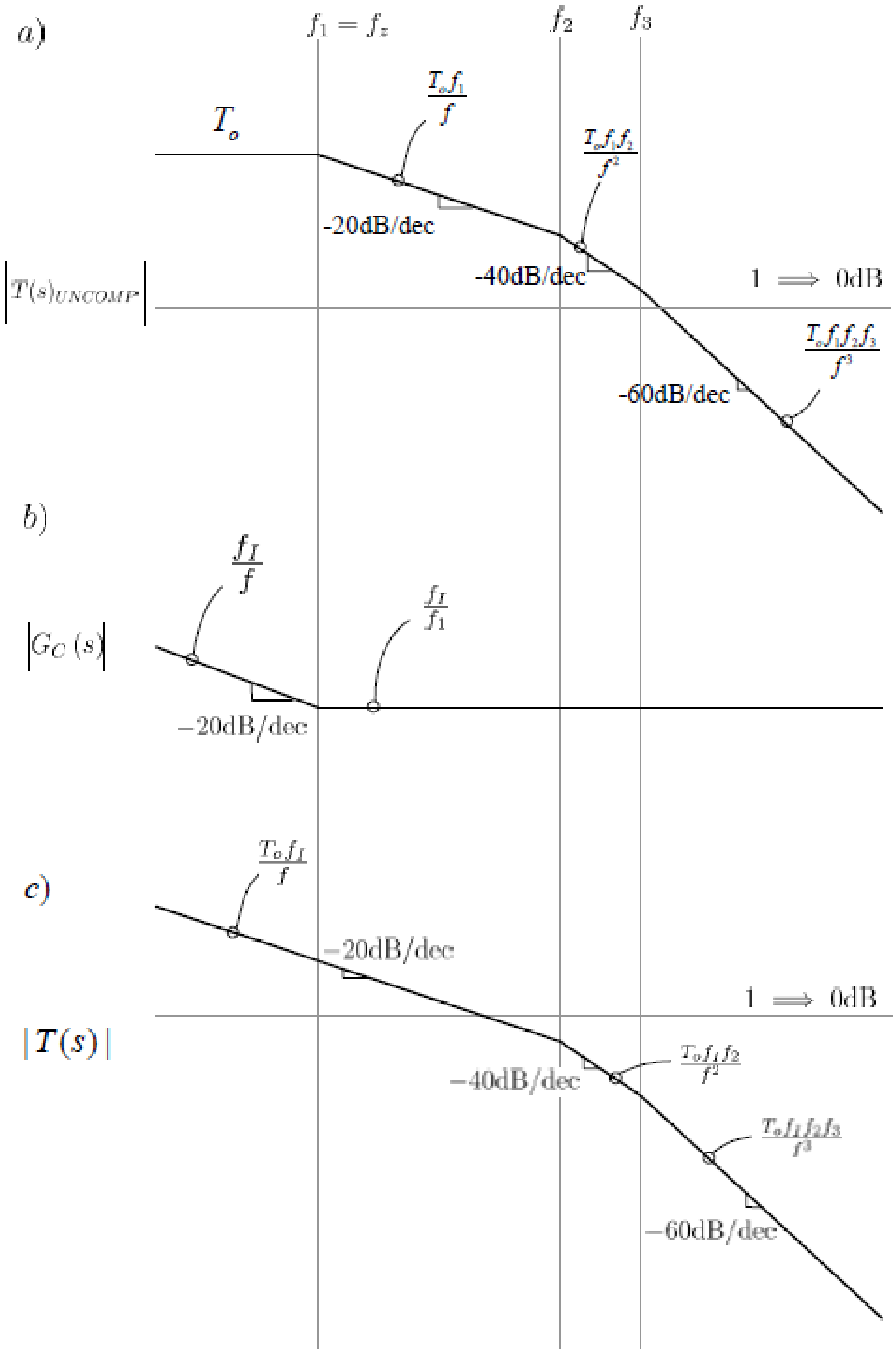

Figure 5.13: Dominant Pole with Zero Magnitude Construction

Modified loop gain response (magnitude and phase):

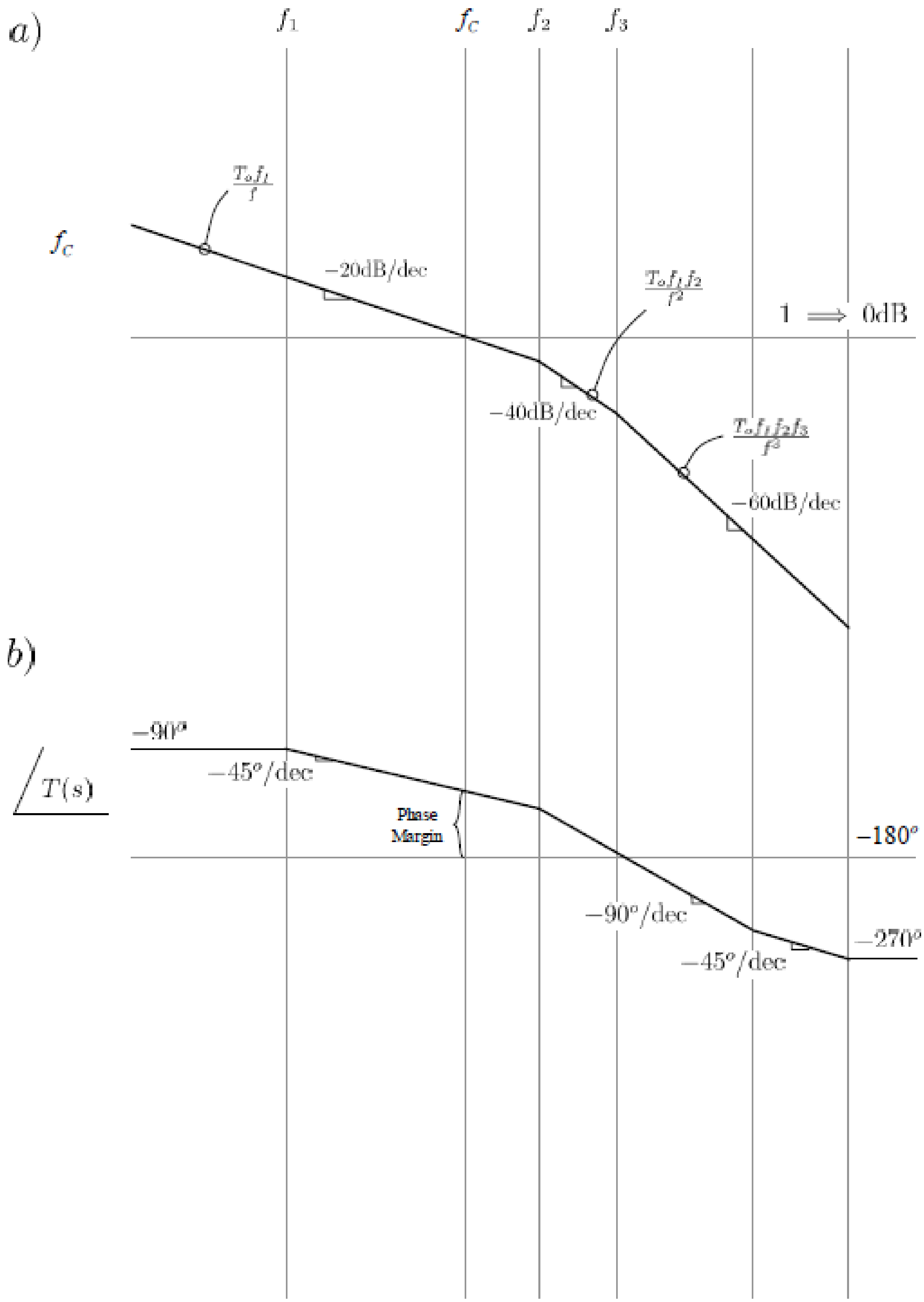

Figure 5.14: Dominant Pole with Zero Compensated System

From the magnitude response we see:

$$\frac{T_o f_I}{f_c} = 1$$

$$\Longrightarrow \quad f_c = T_o f_I \tag{5.16}$$

Phase response at a frequency f is given by:

$$\phi_f = -90 - \arctan\left(\frac{f}{f_2}\right) - \arctan\left(\frac{f}{f_3}\right) \tag{5.17}$$

With $f = f_c$ (the unity gain crossover frequency):

Consequently the phase margin is given by:

$$\begin{aligned} PM &= 180 + \phi_{f_c} \\ &= 90 - \arctan\left(\frac{f_c}{f_2}\right) - \arctan\left(\frac{f_c}{f_3}\right) \\ &= 90 - \arctan\left(\frac{T_o f_I}{f_2}\right) - \arctan\left(\frac{T_o f_I}{f_3}\right) \end{aligned} \tag{5.18}$$

With $PM = 45°$ we find:

$$f_I = 0.258$$

➔

$$\omega_I = 1.623$$

and

$$f_c = 64.58\ Hz$$

Verify with Matlab *margin* command:

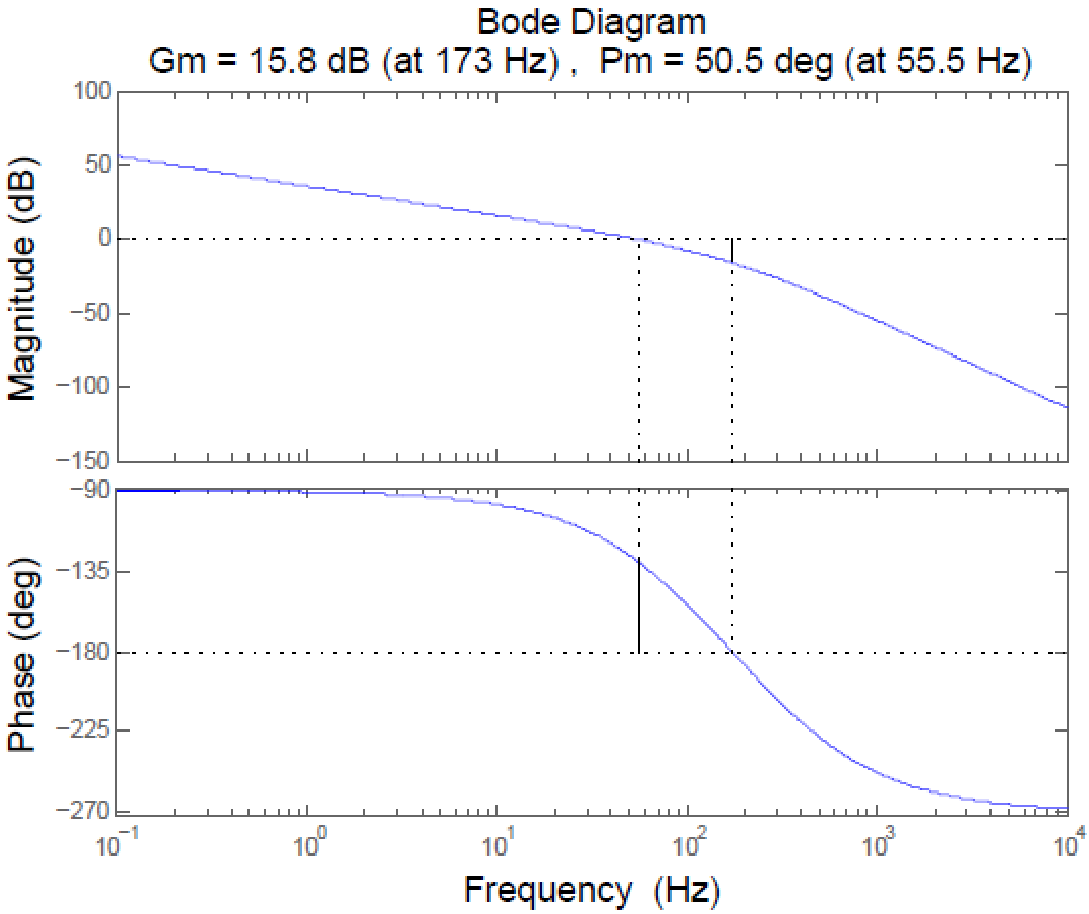

Figure 5.15: Loop Gain and Phase Response of the Dominant Pole Compensated System with Zero

Transient performance:

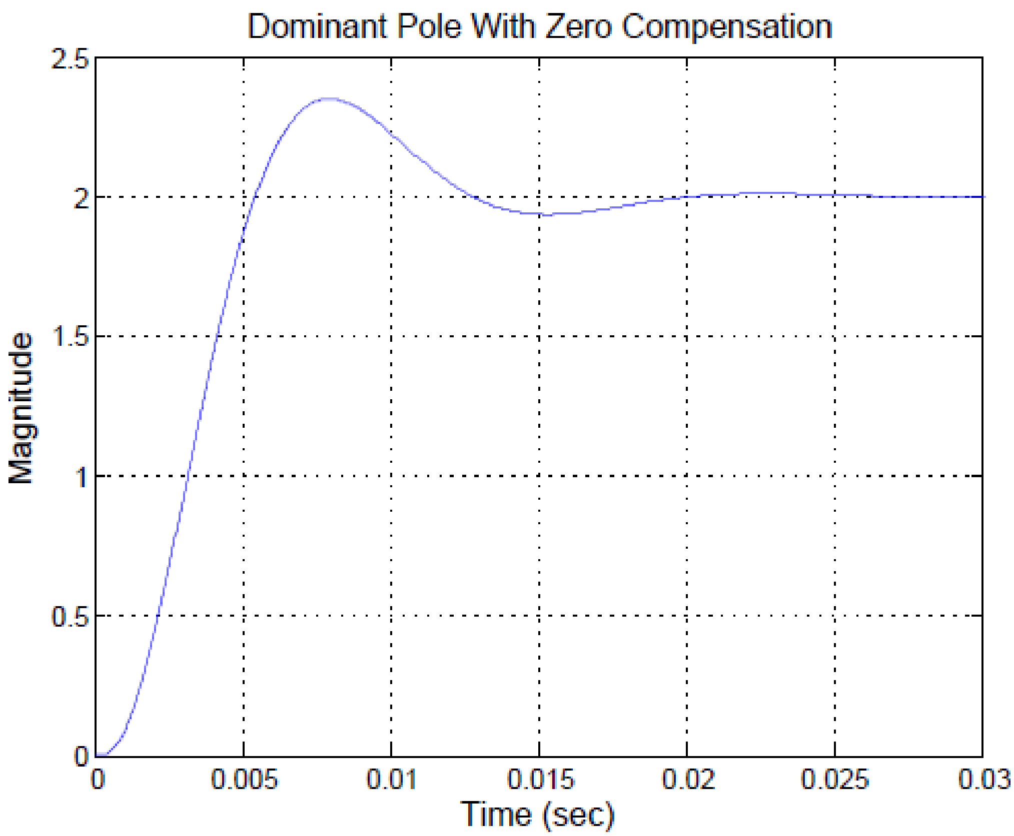

Figure 5.16: Step Response of the Dominant Pole with Zero Compensated System

Table 5.4: Dominant Pole with Zero Compensated System Features

Dominant Pole with Zero Compensation	
Characteristics	Value
Overshoot	22.2%
Rise time	3.05 ms
Settling time	16.7 ms
Steady-state error	0 %
Bandwidth	62.3 Hz
Phase margin	46.3 degree
Gain margin	14.5 dB

Note: speed of response has been improved by the increase of bandwidth.

Next: we will lower the overshoot by increasing the phase margin

Compensator Design #4: Dominant Pole with Zero Compensator, with improved phase margin

Rather than require $PM = 45°$ **we'll redesign for** $PM = 60°$

$$G_c(s) = \frac{\omega_I}{s}\left(1 + \frac{s}{\omega_z}\right)$$

We'll keep $\omega_z = \omega_1$, using formulas from before results in:

$$f_I = 0.1636$$

➔

$$\omega_I = 2\pi f_I = 1.0276$$

Verify with Matlab *margin* command:

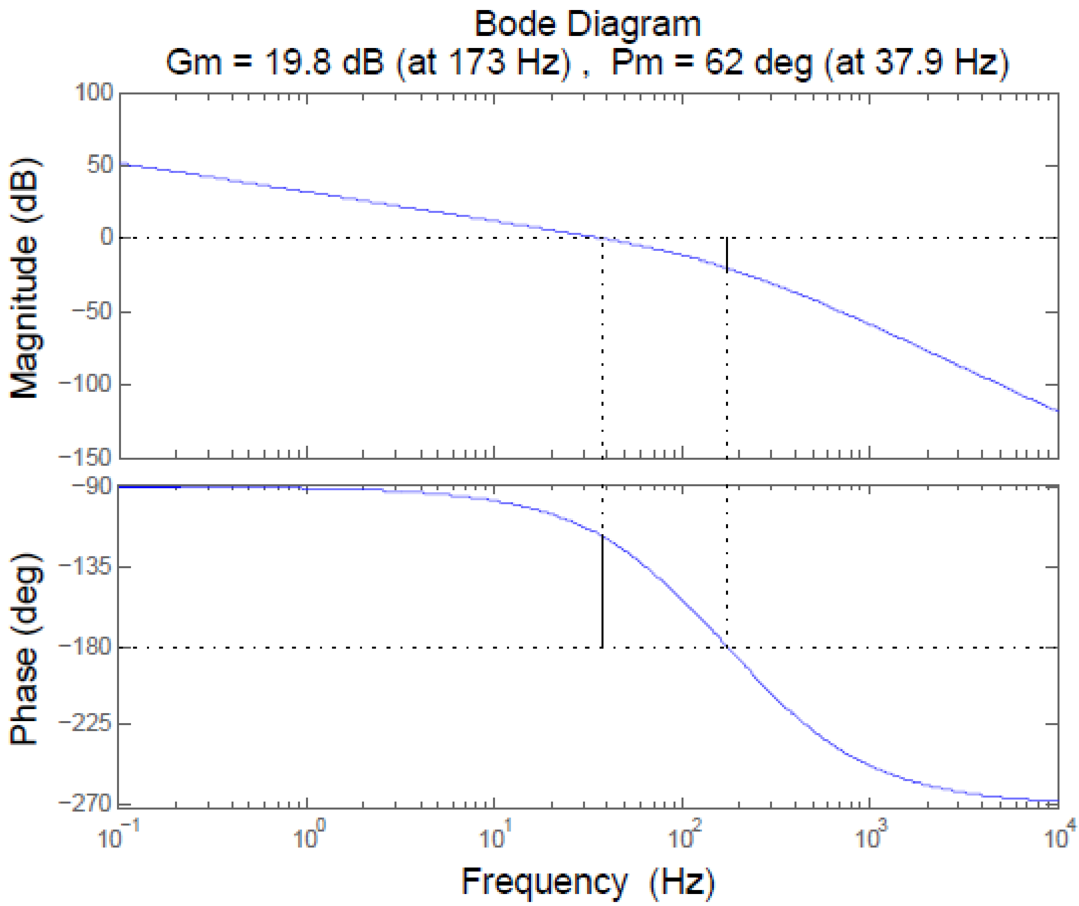

Figure 5.17: Matlab Analysis of Dominant Pole Compensated System with Zero (Improved Margin)

Transient performance:

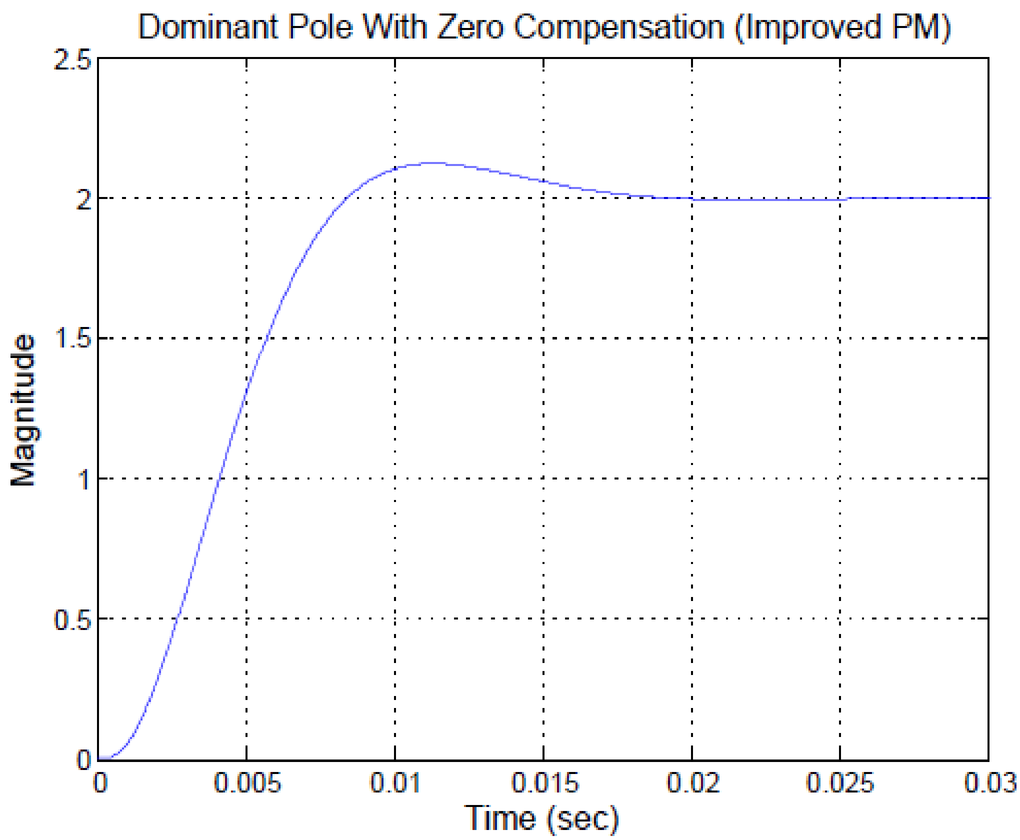

Figure 5.18: Step Response of the Dominant Pole with Zero Compensated System (Improved Margin)

Table 5.5: Dominant Pole with Zero (Improved PM) Compensated System Features

Dominant Pole with Zero Compensation (Improved Margin)	
Characteristics	Value
Overshoot	4.46 %
Rise time	5.81 ms
Settling time	16.5 ms
Steady-state error	0
Bandwidth	35.1 Hz
Phase margin	64 degree
Gain margin	20.6 dB

Note: the overshoot has been reduced to 4.46%

Compensator Design #5: Lead Compensator

$$G_c(s) = G_{c_o}\frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}}, \qquad \omega_z < \omega_p \qquad (5.19)$$

Three parameters need to be determined: $G_{Co},\ \omega_z,\ and\ \omega_p$

➔

Lead Compensated Loop Gain:

$$T(s) = \frac{T_o G_{co}\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_p}\right)\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)}$$

Lead compensator Bode plot:

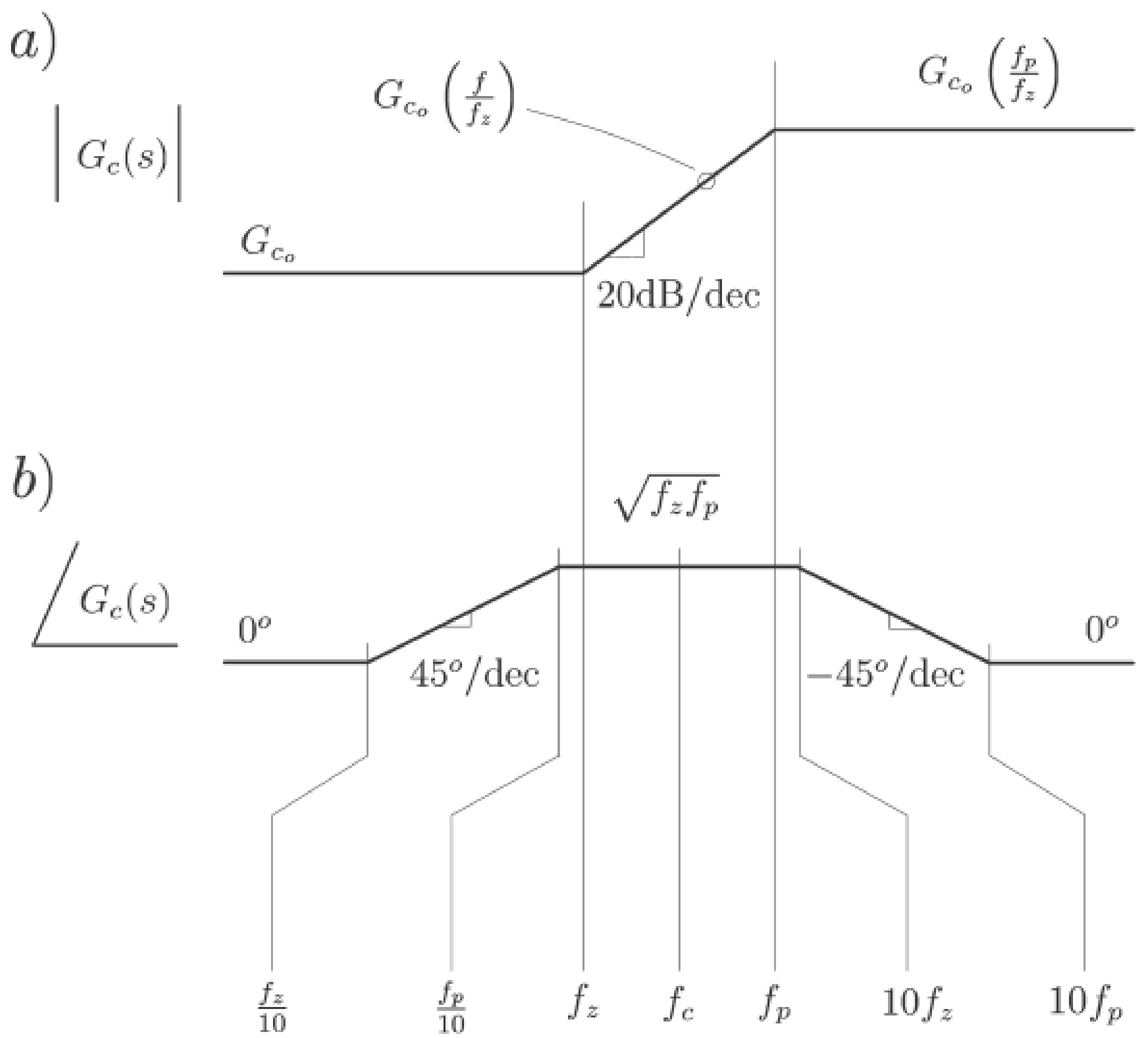

Figure 5.19: Bode Diagram: Lead Compensator

the lead compensator provides a phase boost that is adjustable based on the pole and zero frequencies. The maximum phase boost ϕ_{max} possible is $\phi_{max} = 90°$ and occurs at a frequency $f_{\phi_{max}}$ which is the geometric mean of the zero and pole frequencies of the compensator. The geometric mean of two numbers represents the midpoint between these numbers when represented on a logarithmic scale.

$$f_{\phi_{max}} = \sqrt{f_z f_p} \tag{5.20}$$

Modified loop gain magnitude response:

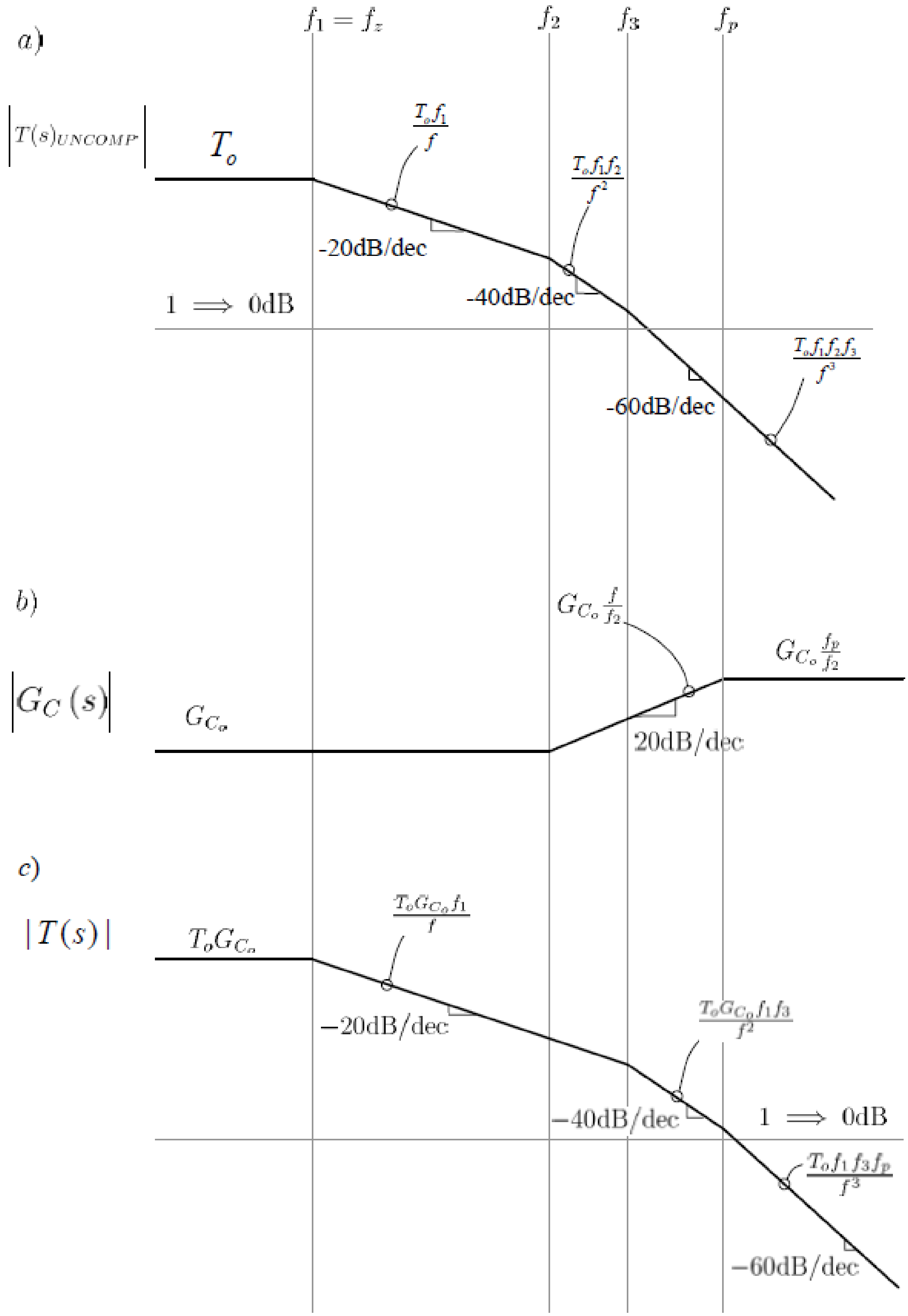

Figure 5.20: Lead Compensation Magnitude Construction

Modified loop gain Bode response (magnitude and phase):

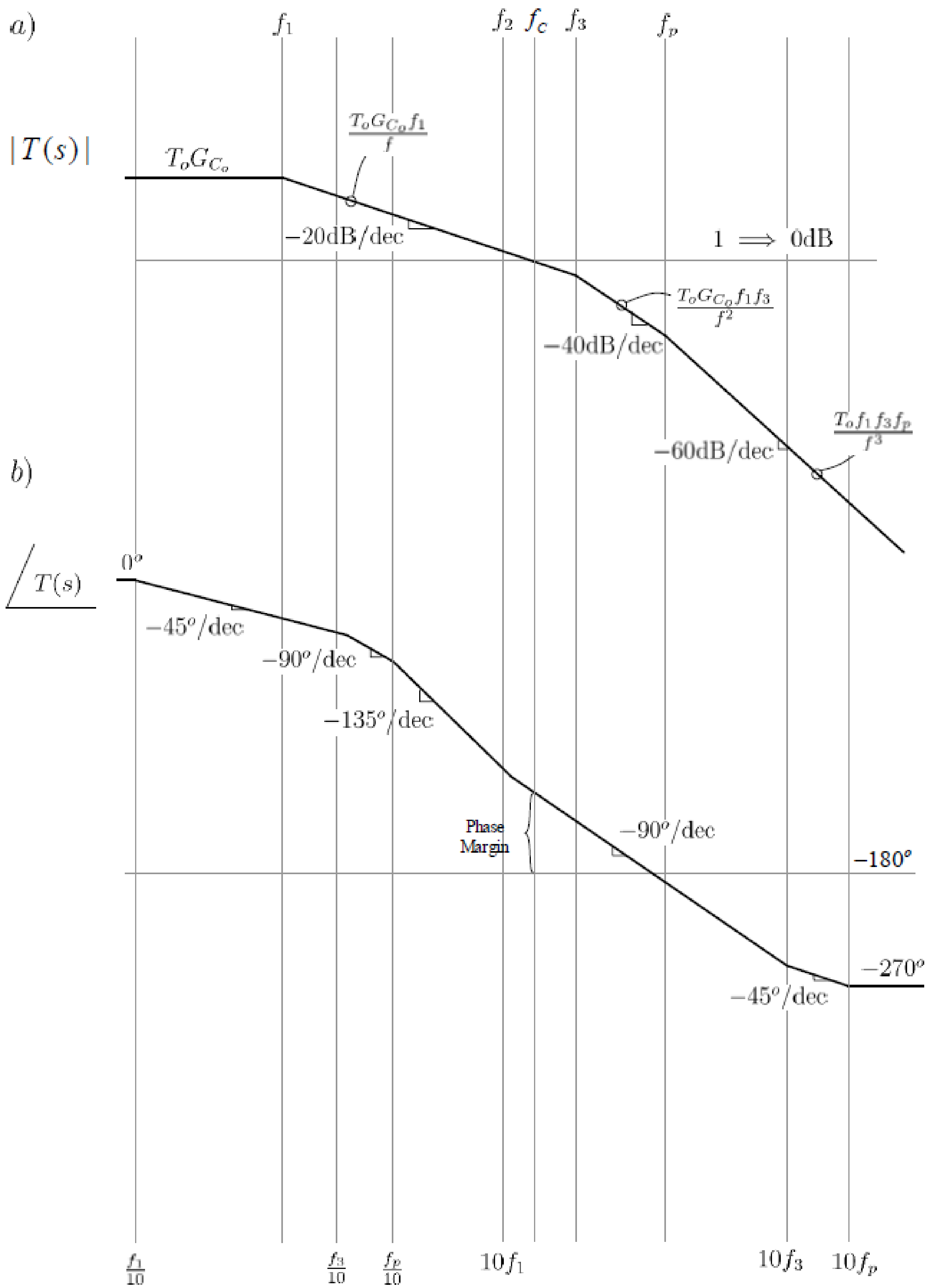

Figure 5.21: Lead Compensated System

Lead Compensated Loop Gain:

$$T(s) = \frac{T_o G_{co}\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_p}\right)\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)}$$

➔

Exact phase ϕ_f at frequency *f* is:

$$\phi_f = \arctan\left(\frac{f}{f_z}\right) - \arctan\left(\frac{f}{f_p}\right) - \arctan\left(\frac{f}{f_1}\right) - \arctan\left(\frac{f}{f_2}\right) - \arctan\left(\frac{f}{f_3}\right) \qquad (5.21)$$

Consequently the phase margin is given by:

$$\begin{aligned} PM &= 180 + \phi_{f_c} \\ &= 180 + \arctan\left(\frac{f_c}{f_z}\right) - \arctan\left(\frac{f_c}{f_p}\right) - \arctan\left(\frac{f_c}{f_1}\right) \\ &\quad - \arctan\left(\frac{f_c}{f_2}\right) - \arctan\left(\frac{f_c}{f_3}\right) \end{aligned} \qquad (5.22)$$

Set

$$\begin{aligned} f_c &= f_{\phi_{max}} \\ &= \sqrt{f_z f_p} \end{aligned} \qquad (5.23)$$

and

In order to minimize the effect on the phase margin of the phase lag due to the compensator pole we will set this pole frequency an order of magnitude above the crossover frequency:

$$f_p = 10 f_c \qquad (5.24)$$

➔

$$f_p = 100 f_z \tag{5.25}$$

And with $f_z = f_2$ and $PM = 60°$ ➔ $f_c = 187\ Hz$

From the magnitude asymptote we see that

$$\frac{T_o G_{c_o} f_1}{f_c} = 1 \tag{5.26}$$

➔

$$G_{c_o} = \frac{f_c}{T_o f_1} \tag{5.27}$$

➔

$$G_{c_o} = 0.0749$$

Lead compensator three parameters:

$$G_{Co} = 0.0749$$

$$\omega_z = 2\pi(100)\ rds/s$$

$$\omega_p = 2\pi(10000)\ rds/s$$

Verify with Matlab *margin* command:

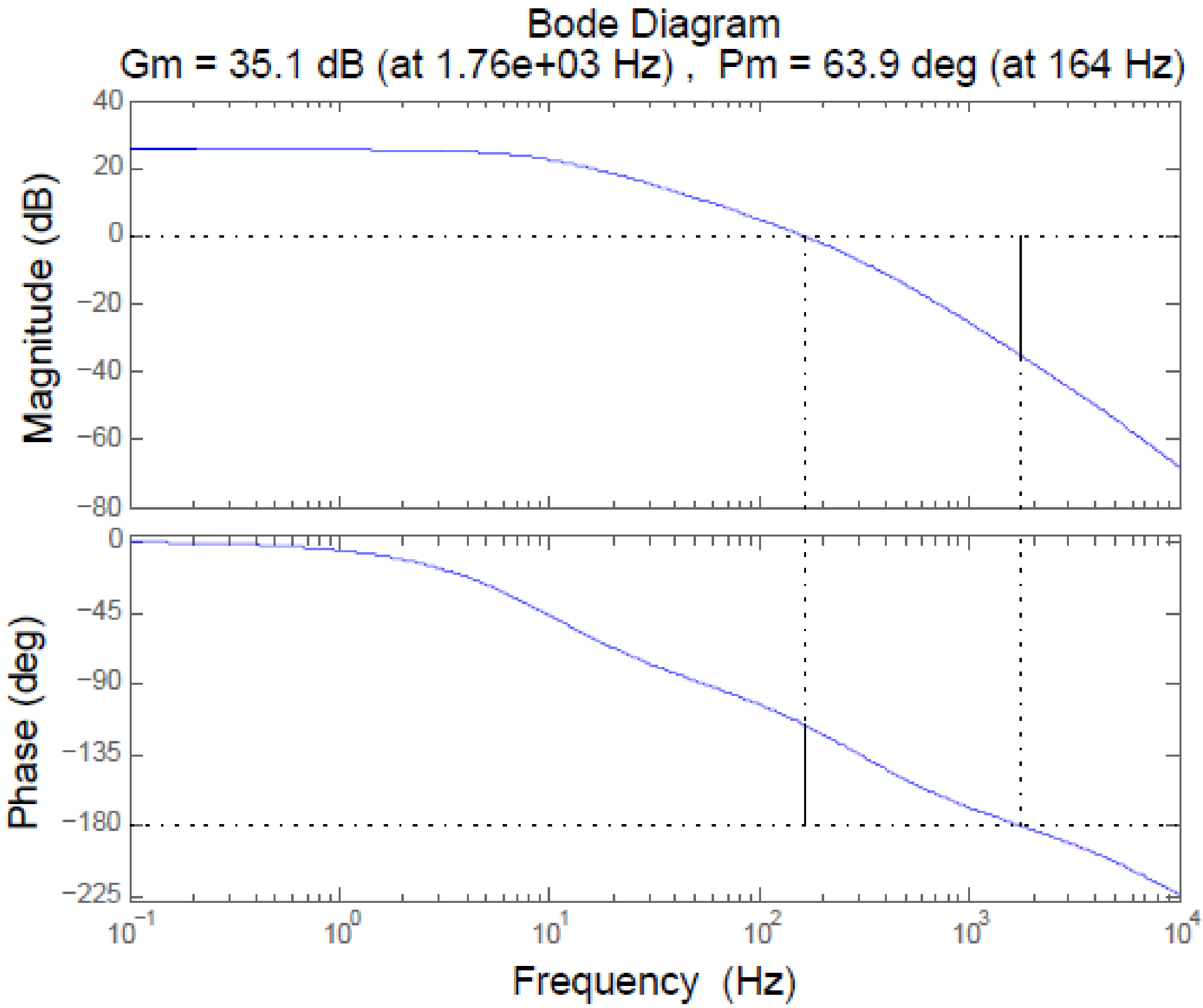

Figure 5.22: Matlab Analysis of Lead Compensated System

Transient performance:

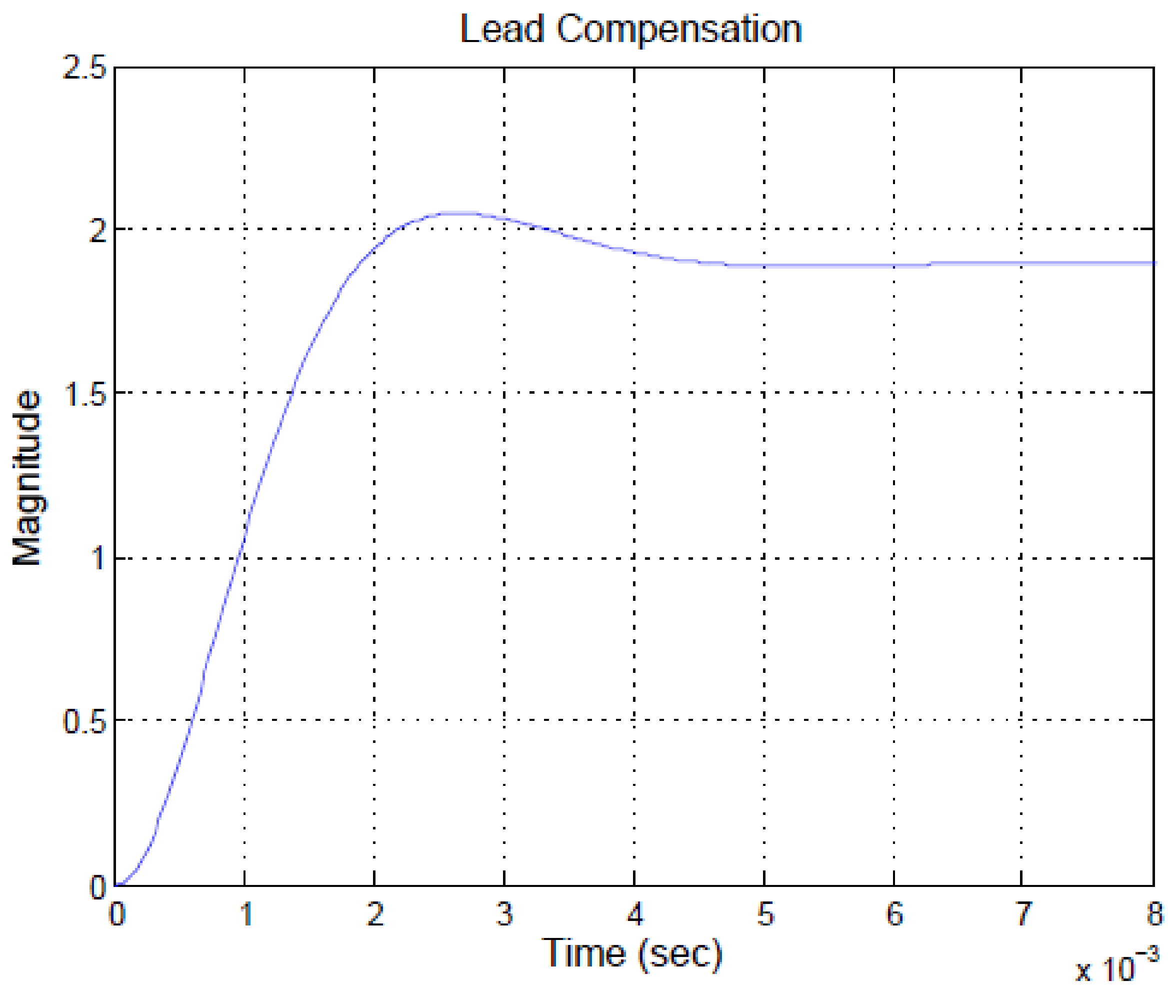

Figure 5.23: Step Response of the Lead Compensated System

Table 5.6: Lead Compensation

Lead Compensation	
Feature	Value
Overshoot	4.23 %
Rise time	2.25 ms
Settling time	6.29 ms
Steady-state error	10%
Bandwidth	83.8 Hz
Phase margin	71.2°
Gain margin	19.3 dB

Note: a non-zero steady state error exists

Compensator Design #6: Lead Compensated System with integrator and zero

$$G_c(s) = \frac{\omega_I\left(1+\frac{s}{\omega_{z_1}}\right)\left(1+\frac{s}{\omega_{z_2}}\right)}{s\left(1+\frac{s}{\omega_p}\right)} \qquad (5.28)$$

To obtain zero steady state error we add an integrator to the previous lead compensator design.

The parameters ω_{z_2} and ω_p correspond to ω_z and ω_p of the lead compensator design, which leaves ω_I and ω_{z_1} to be determined. ω_{z_1} can be simply set to ω_1. The low frequency gain of the lead compensator of the previous section was denoted G_{c_o}. This was the value of the loop magnitude at f_1 (in particular, and below this frequency, in general). To maintain this value of gain at f_1 we will adjust ω_I to achieve this. The low frequency magnitude asymptote is given by $\frac{f_I}{f}$ so that at f_1 we have

$$\frac{f_I}{f_1} = G_{c_o} \implies f_I = G_{c_o} f_1 \qquad (5.29)$$

This completes the design of this compensator.

Verify with Matlab *margin* command:

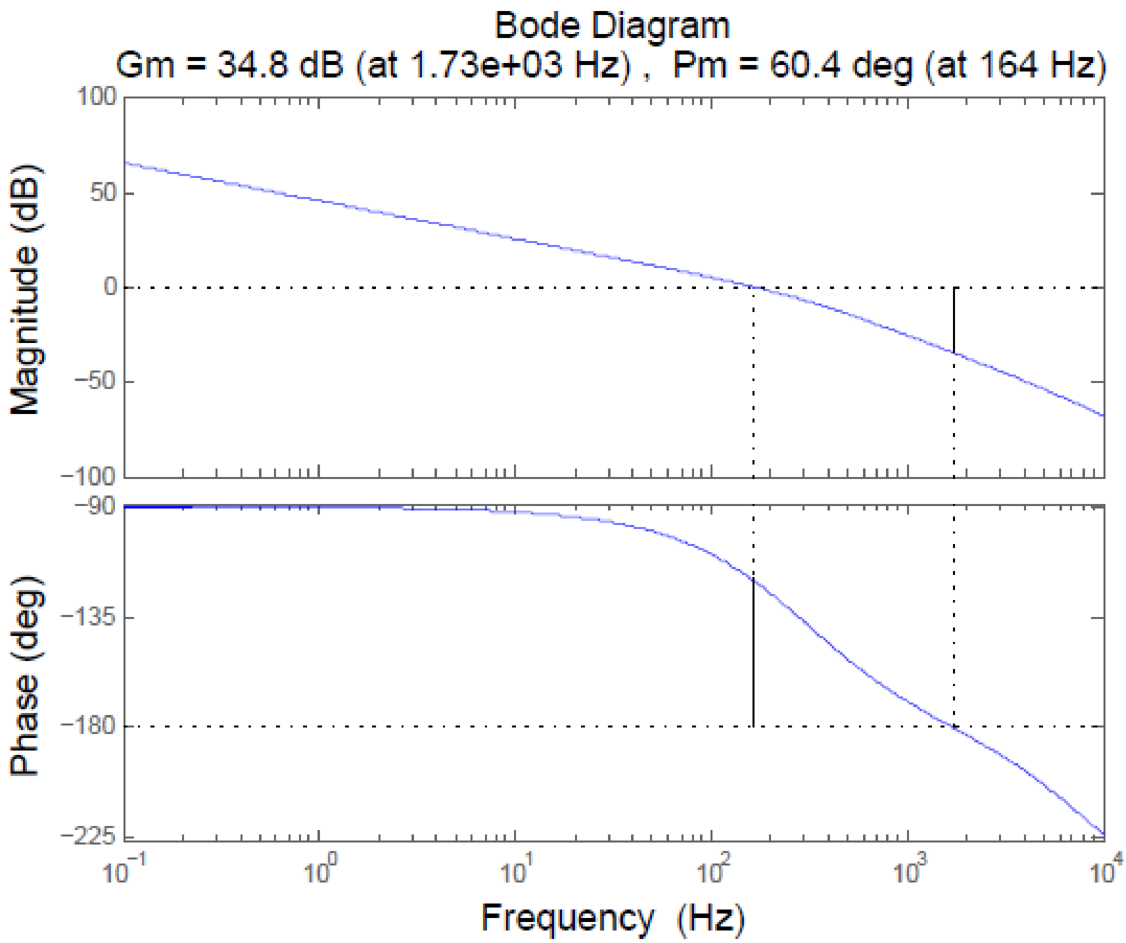

Figure 5.24: Matlab Analysis of Lead Compensated System with integrator and zero

Transient performance:

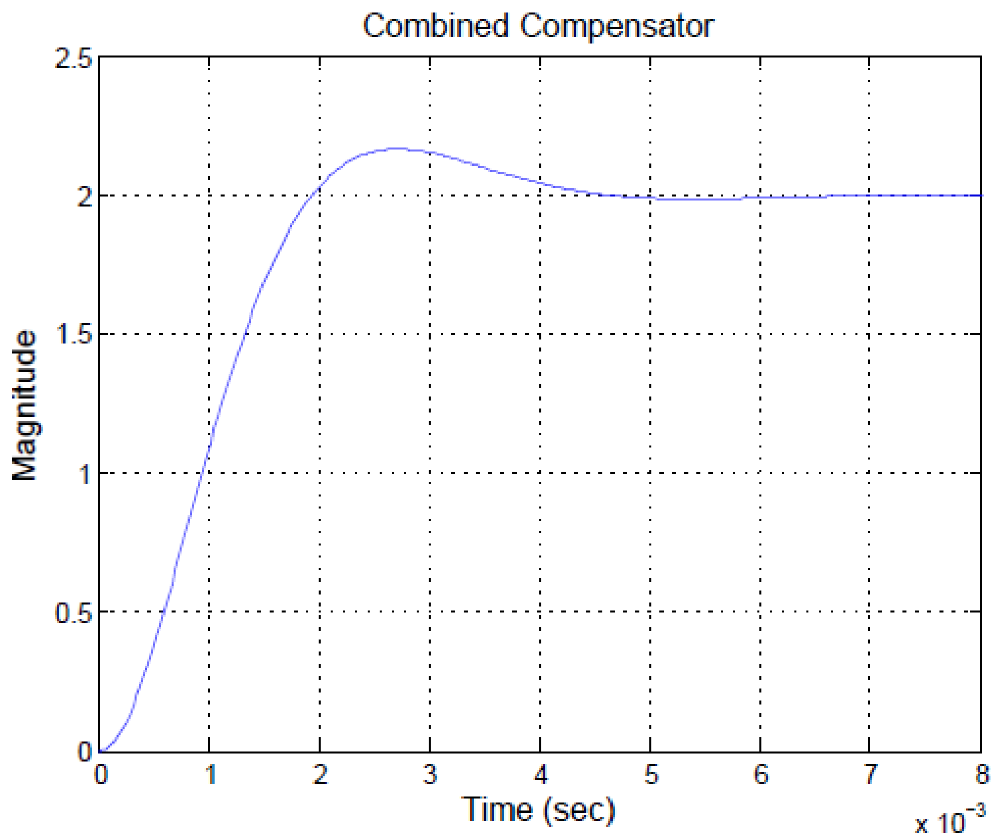

Figure 5.25: Step Response of the Lead Compensated System with integrator and zero

Table 5.7: Lead Compensation with integrator and zero

Lead Compensation with integrator and zero	
Feature	Value
Overshoot	4.23 %
Rise time	2.25 ms
Settling time	6.29 ms
Steady-state error	10%
Bandwidth	83.8 Hz
Phase margin	71.2°
Gain margin	19.3 dB

Table 5.8: Summary of Compensators

$$G(s) = \frac{G_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)}$$

$$H(s) = k$$

where $G_o = 500$, $\omega_1 = 2\pi(10)$, $\omega_2 = 2\pi(100)$, $\omega_3 = 2\pi(300)$, and $k = 0.5$.

$G_c(s)$	$G_c(s)$ parameters	f_c (Hz)	ϕ_{PM} (deg)	GM (dB)	OS (%)	t_r (ms)	t_s (ms)	ERR (%)
1 (uncompensated)	none	385	-36	-15	na	na	na	∞
k_p	$k_p = 0.03$	63	55	15	20	2.9	15.4	-11
$\frac{\omega_I}{s}$	$\omega_I = 0.25$	7.8	46	18	22	25	137	0
$\frac{\omega_I}{s}\left(1+\frac{s}{\omega_z}\right)$	$\omega_I = 1.62$ $\omega_z = 2\pi(10)$	56	51	16	17	34	18	0
$\frac{\omega_I}{s}\left(1+\frac{s}{\omega_z}\right)$	$\omega_I = 1.03$ $\omega_z = 2\pi(10)$	38	62	20	6	5.3	16	0
$G_{c_o}\frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}}$	$G_{c_o} = 0.075$ $\omega_z = 2\pi(100)$ $\omega_p = 2\pi(10,000)$	164	64	35	8.1	1.3	3.9	-5
$\frac{\omega_I\left(1+\frac{s}{\omega_{z_1}}\right)\left(1+\frac{s}{\omega_{z_2}}\right)}{s\left(1+\frac{s}{\omega_p}\right)}$	$\omega_I = 4.71$ $\omega_{z_1} = 2\pi(10)$ $\omega_{z_2} = 2\pi(100)$ $\omega_p = 2\pi(10,000)$	164	60	35	8.3	1.3	4	0